Relative portfolio risk decomposition and attribution

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Abstract

In a related paper, see reference [4], we compute the risk decomposition of a portfolio with respect to a generic homogeneous risk measure. In this paper we extend those results to the risk decomposition of an active portfolio: a portfolio measured against a benchmark. We find that all the results obtained for the portfolio can be extended to the active portfolio. Furthermore, analogously to a similar analysis in portfolio performance attribution, we split active risk according to the investment decision process and obtain a selection risk, an allocation risk, and interaction risk. Finally, we consider risk decomposition and attribution for portfolio with assets in different currencies splitting risk into local components and a currency-exchange component.

Keywords: risk, active risk, risk contributions, risk attribution, risk components, portfolio risk, risk measures, value at risk, var, risk decomposition, marginal risk, portfolio segments, relative risk, benchmarks, risk attribution, currency-exchange risk contribution

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1 Introduction

Even thought the asset-management industry has undergone many changes in recent years, it is still possible to split managed funds into two broad categories: absolute-return funds and benchmark funds. In the first category, the absolute return funds, we place all the funds that try to create alpha (i.e. pure return, this is the reason why sometimes these funds are referred to as alpha funds), where the investor does not have any other measurement of performance if not that of the fund itself (think the reference benchmark to be the risk free rate). In this set, for example, we find most hedge funds. In the second category, the benchmarked funds, investors expect a return, or a loss, similar to that of an established benchmark portfolio (sometimes simply named benchmark). In this case the asset manager performance is measured on how much better, or worse, she performs with respect to the reference benchmark portfolio. Hence if the fund makes a loss, e.g. 10%, which is smaller than the benchmark loss, e.g. 15%, the fund manager will still have done better than the benchmark and have, therefore, generated an active positive return.

As most investors find out fund performance is good only if the risk taken to obtain it is appropriate. In the case of benchmark funds risk should be compared with respect to the benchmark portfolio so that that the interest is in how much risk we are taking that was not originally in the benchmark.

In literature, see for example [3], very often one measures the volatility of the active portfolio (a.k.a. the tracking error), i.e. a virtual book created by buying the reference portfolio and by selling the benchmark portfolio. We will refer to this virtual book interchangeably with the term active portfolio and relative portfolio and call its risk both active risk or relative risk.

In this paper we analyze active risk and ask ourselves how to split it into additive risk components, the components that will sum to the relative risk, so that each component can be associated to a single security, an investment sector, or an asset class. The literature on this subject is minimal: in the paper of Garman, see reference [2], value at risk (sometimes shortened as VaR) is decomposed for the variance/covariance model. Here we will use the same approach of reference [4] applied to the active portfolio.

After briefly introducing the paper notation in section 2, we proceed, in section 3, to compute the relative risk and its additive decomposition. In sections 4 and 5 we consider the allocation process more in details: we assume that the decisions of sector allocation and stock picking are taken separately and compute the appropriate risk for each process. In section 6 we discuss portfolios containing assets in different currencies and how to decompose their risk into a local-currency component and a currency-exchange component. Risk-attribution results are also split in local and currency-exchange components.

Finally, in order to keep the paper as simple as possible, we will limit the discussion of this paper only to single-level decomposition and attribution.

2 General definitions and notation

In this section we introduce the paper notation and some basic definitions. The reader already familiar with the definitions and the notation of reference [4] should recognize many similarities. Furthermore, in this paper the main feature is the presence of two main portfolios: the original portfolio and the benchmark portfolio.

2.1 Asset universe and portfolio future value

Consider a universe of $n$ assets allowed to be bought or sold in order to build a portfolio. Suppose that each asset, denoted by an index $i$ with $i = 1, \ldots, n$, has an outstanding value of $W_i$ in the portfolio, so that the initial portfolio value is given by

$$W = W_1 + \ldots + W_n.$$  \hspace{1cm} (1)

Since not all assets in the universe might be present in the portfolio, we set $W_i=0$ for all assets that are not part of the portfolio. In order to simplify the discussions that follow we consider only assets for which the
exposure to risk is the same as its outstanding value. In any case, assets for which this is not true can always be decomposed into two or more legs, each of which has this property. Assuming a positive portfolio outstanding value $W$, for each asset $i$ we define the asset weight $w_i$ in the portfolio as,

$$w_i = \frac{W_i}{W}.$$

(2)

As a consequence of the dynamics of financial markets, after a certain period of time, usually chosen to be one day, each asset $i$ will have an outstanding value $u_iW_i$, so that its percentage return over the period will be given by,

$$R_i^\% = \frac{u_iW_i - W_i}{W_i} = u_i - 1.$$

(3)

Since we assume there is no re-balancing of the portfolio during the evolution period, the portfolio value after the period will be given by,

$$P = u_1W_1 + \ldots + u_nW_n.$$

(4)

Notice that whilst the values of the $W_i$'s are well known before and after the period, the quantities $u_i$'s are not known and can be considered random variables.

2.2 Benchmark associated to a portfolio

Consider a benchmark associated to the portfolio containing assets also chosen from the same universe (we can always enlarge the universe to contain all the benchmark assets). Denote the outstanding value of asset $i$ in the benchmark with the symbol $W_i^b$. We also assume that initially the portfolio and its benchmark have the same total outstanding value, i.e.,

$$W = W_1^b + \ldots + W_n^b.$$

(5)

Commonly the quantities $W_i^b$'s are obtained from a set of benchmark weights $w_i^b$'s as,

$$W_i^b = w_i^bW.$$

(6)

where, for every $i$, $w_i^b$ is the percentage weight of asset $i$ in the benchmark (or zero if the asset is not in the benchmark). The benchmark weights must sum to 100%, i.e.,

$$1 = 100\% = w_1^b + \ldots + w_n^b.$$

(7)

The financial-market dynamics will result in a change of the benchmark value after the given time period. Since we assume that there is no re-balancing during this period, the benchmark future value will be given by

$$B = u_1W_1^b + \ldots + u_nW_n^b.$$

(8)

In general we have $P \neq B$ and the difference between the portfolio and the benchmark value is the quantity of interest. As already mentioned in the introduction, we define the active, or relative, portfolio as the portfolio in which asset $i$ has an outstanding value $W_i'$,

$$W_i' = W_i - W_i^b.$$

(9)

Since at the initial time portfolio and benchmark have the same value the relative portfolio is initially worthless. However, after the evolution period has passed the relative portfolio value will be given by,

$$P' = u_1W_1' + \ldots + u_nW_n' = u_1(W_1 - W_1^b) + \ldots + u_n(W_n - W_n^b) = P + \delta,$$

which is, in general, not zero.

**Basic risk-decomposition definitions** The relative-risk analysis is based on the definition of active portfolios, therefore, we define the relative risk $R'$ as the risk incurred in holding the portfolio with a future value $P'$. As already mentioned in section 1, the aim of relative risk decomposition is to find $n$ components $C_i'$'s, one for each asset in the universe, so that their sum is the relative risk, i.e.,

$$R' = \mathcal{R}(P') = C_1' + \ldots + C_n',$$

(11)

where $\mathcal{R}$ is the function that maps the weights of $P'$ into the active risk $R'$. Examples of risk functions $\mathcal{R}$ will be given later in section 3.
Finally, we define marginal relative risk as the change of relative risk for a small increase in the holding of an asset, i.e.,

$$M_{ri} = \frac{\partial R}{\partial W_i},$$  \hspace{1cm} (12)$$

for each \(i = 1, \ldots, n\).

### 2.3 Segment outstanding values and their weights

As explained in reference [4], we partition the universe assets according to a given attribute (e.g. country) and define the segments, denoted by \(a\), \(b\), \(c\), and so on, so that each asset is in one, and only one, segment.

We can arrange the terms in equation (1) so that the total portfolio outstanding value \(W\) is given by the sum of the segment outstanding values, i.e.,

$$W = W_a + W_b + W_c + \ldots,$$  \hspace{1cm} (13)$$

where for each segment, for example segment \(a\), we have,

$$W_a = \sum_{i \in a} W_i.$$  \hspace{1cm} (14)$$

The notation \(i \in a\) under the sum symbol \(\sum\) means that the sum is performed over all assets in sector \(a\). Also we can split the total portfolio value according to the segment outstanding value in the benchmark, i.e.,

$$W = W_a^b + W_b^b + W_c^b + \ldots,$$  \hspace{1cm} (15)$$

where for each segments, for example segment \(a\), we defined

$$W_a^b = \sum_{j \in a} W_j^b.$$  \hspace{1cm} (16)$$

### 3 Decomposition of relative risk

In this section we apply the results of reference [4] to the active portfolio in order to compute its decomposition and its marginal risks. In this section we will initially derive the risk-decomposition results for special cases of the risk function \(R\) and later generalize those results.

#### 3.1 Covariance framework for relative risk

In the covariance framework relative risk is defined as the standard deviation of the future portfolio value \(P'\). The active-portfolio variance can be computed as,

$$\sigma^2 = \sum_{i,j} \Gamma_{ij} (W_i - W_i^b) (W_j - W_j^b).$$  \hspace{1cm} (17)$$

where the covariance matrix \(\Gamma\) is defined as,

$$\Gamma_{ij} = \langle (u_i - \bar{u}_i) (u_j - \bar{u}_j) \rangle.$$  \hspace{1cm} (18)$$

The corresponding risk measure, the tracking error, is defined as the square root of \(\sigma^2\), hence,

$$R' = R(P - B) = \sqrt{\sigma^2} = \frac{\sigma^2}{\sigma},$$  \hspace{1cm} (19)$$

where in the last equation we used the definition,

$$M_{ri}' = \sum_{j=1}^n \frac{\Gamma_{ij}}{\sigma} (W_j - W_j^b).$$  \hspace{1cm} (20)$$

Computing the derivative of \(\sigma^2\) from equation (17), we can easily show that the numbers \(M_{ri}'\)’s are the marginal relative risks for the tracking error and satisfy equation (12). Furthermore the result shown in equation (19) is the first known example of decomposition of active risk (tracking error in this case) and it has been widely used in portfolio risk analysis for many years (see, e.g., reference [3]).

#### 3.2 Decomposition in a generic simulation framework

We derive in the following paragraphs a result very similar to that derived in subsection 3.1 for a generic simulation framework.

In a generic simulation framework the values of the \(u_i\)’s are sampled from computer simulations. Hence given \(N\) simulations, indexed by \(s = 1, \ldots, N\), for each asset \(i\) we denote by \(u_i^s (W_i - W_i^b)\) the asset future
value in the active portfolio. The relative future portfolio value \( P'_s \) will therefore be given by,

\[
P'_s = P^s - B^s = u^s (W_1 - W^b_1) + \ldots + u^s_n (W_n - W^b_n) .
\]

(21)

The collection of all \( P'_s \)'s is then a numerical representation of the expected future distribution of \( P' \) and, on this simulated distribution, it is possible to compute various risk measures.

In order to compute the relative value at risk, relative VaR or rVaR for short, consider a percentile \( c \), then the relative value at risk is the loss incurred in the simulation \( s \) so that \( c \) \( N \) scenarios have a loss less than that of \( s \). Formally, given \( s \) so that,

\[
\text{Prob}(P^s - B^s \leq P^s - B^s) = 1 - c
\]

(22)

and

\[
\text{Prob}(P^s - B^s > P^s - B^s) = c ,
\]

(23)

the relative value at risk, denoted by \( R^{\text{VaR}}_r \), is defined as,

\[
R^{\text{VaR}}_r = \max (- [P^s - B^s], 0) .
\]

(24)

Now consider the common case in which \( R^{\text{VaR}}_r > 0 \), using equation (24) we write,

\[
R^{\text{VaR}}_r = C'_1 + \ldots + C'_n ,
\]

(25)

where,

\[
C'_i = (1 - \bar{u}_1) (W_i - W^b_i) , \quad \ldots \quad C'_n = (1 - \bar{u}_n) (W_n - W^b_n) .
\]

(26)

and, for each \( i \), we defined \( \bar{u}_i = u^s_i \).

The split of relative VaR as outlined in equation (25) is known as the relative risk decomposition of the portfolio relative to its benchmark. We denote with \( C'_i \) the relative risk component for each asset with index \( i \), as defined in equation (26).

3.3 Average value at risk and its decomposition

Similarly to the discussion of reference [4] we expect the decomposition of relative value at risk, defined in equation (26), to be unstable.

In order to obtain a robust definition for the relative-risk components we need to introduce the concept of average relative value at risk. Hence, given a lower percentile \( \hat{c} \) and an upper percentile \( \hat{c} \), with \( \hat{c} < \hat{c} \), the average relative value at risk, denoted by the symbol \( R^{\text{aVaR}}_r \), is defined as the average losses in the relative portfolio for all simulations which fall between the percentiles \( \hat{c} \) and \( \hat{c} \). Formally, define \( S \) as the set of all simulations \( s \) so that,

\[
P'_s \leq P'_s \leq P'_s ,
\]

(27)

where \( s \) and \( s \) are, respectively, the scenarios associated with the percentiles \( \hat{c} \) and \( \hat{c} \) (defined by equation (23)). The average relative value at risk is then given by,

\[
R^{\text{aVaR}}_r = \max \left[ \frac{1}{N_s} \sum_{s \in S} (B^s - P^s), 0 \right] .
\]

(28)

We refer to paper [4] for handling the case in which \( N(1 - \hat{c}) \) or \( N(1 - \hat{c}) \) are not integers.

It is easy to show that the relative average VaR can be decomposed as,

\[
R^{\text{aVaR}}_r = \sum_{i=1}^{n} C^r_i ,
\]

(29)

with,

\[
C^r_i = (W_i - W^b_i) \frac{1}{N_s} \sum_{s \in S} (1 - u^s_i) .
\]

(30)
The components just defined do not suffer (or, rather, they suffer less) from the stability issues associated with the components defined in equations (26).

Choice of upper and lower percentiles The discussion of reference [4] about the possible different choices of the upper and lower percentiles \( \hat{c} \) and \( \check{c} \) applies to the current case of relative risk decomposition. Therefore, we advise the reader to refer to that paper as a guide for the correct choice of percentiles. We recall here that, depending on the choice of the upper and lower percentiles, one can obtain a relative risk decomposition for expected shortfall, a symmetric average VaR, or an unbiased average VaR.

3.4 Decomposition for a generic risk function

In this section we derive the generic formula for relative-risk decomposition. As shown in reference [4] we consider a smooth risk function homogeneous in the relative outstanding values \( W' \) defined by equation (9). Because of Euler’s theorem any generic risk function \( R \) homogeneous in \( W' \) can be written as

\[ R(W'_1, \ldots, W'_n) = W'_1 \frac{\partial R}{\partial W'_1} + \ldots + W'_n \frac{\partial R}{\partial W'_n}. \]  

(31)

Since \( W'_i = W_i - W^b_i \), we have,

\[ \frac{\partial R}{\partial W'_1} = \frac{\partial R}{\partial W'_1}, \]  

(32)

so that,

\[ R = (W_1 - W^b_1) \frac{\partial R}{\partial W'_1} + \ldots + (W_n - W^b_n) \frac{\partial R}{\partial W'_n}. \]  

(33)

Therefore, similarly to the results described in reference [4], we can show that it is always possible to decompose relative risk as,

\[ R'_r = C'_r + \ldots + C'_n, \]  

(34)

with

\[ C'_r = M'_r (W_i - W^b_i) , \]  

(35)

and

\[ M'_r = \frac{\partial R}{\partial W'_r}. \]  

(36)

In view of the simplicity of this derivation, completely based on Euler’s theorem, we stress once more the importance of this result: given a smooth homogeneous risk function it is always possible to find a relative-risk decomposition with relative-risk components given by equation (35).

3.5 Sector aggregation of components and marginal relative risk

We now consider relative-risk decomposition at sector level. Given a certain attribute we split the universe in segments \( a, b, c, \) and so on, so that each asset is in one and only one segment. The aggregate value of the active portfolio for each segment, for example segment \( a \), is given by,

\[ W'_a = W_a - W^b_a, \]  

(37)

Rearranging the terms in equation (33), we can write,

\[ R'_r = C_a + C_b + \ldots, \]  

(38)

where for each sector, for example sector \( a \), the relative risk component is given by,

\[ C_a = \sum_{i \in a} C_i. \]  

(39)

Inspired by equation (35), we can define the marginal relative risk for all sectors, for example sector \( a \) for which \( W'_a \neq 0 \), as the quantity \( M'_a \) satisfying,

\[ C'_r = M'_a W'_a. \]  

(40)

The number \( M'_a \) is the sector-\( a \) marginal relative risk and can be interpreted as the change of relative risk \( a \) for an increase of one unit of currency uniformly in sector \( a \).

4 Risk attribution and allocation risk

4.1 Risk Attribution

It is well known, see for example reference [1], how to compute the selection and allocation effects for portfolio performance. In this and in the following sections...
we extend these concepts to risk and define a risk attribution: a set of risk figures that split risk to the different subjects involved in the allocation process. Therefore, given an attribute that splits the universe into segments, the goal of risk attribution is to split the relative portfolio risk into:

Allocation risk the active risk coming from sectors having a different weight in the portfolio and in the benchmark

Selection risk the active risk coming from stock picking, i.e. the risk resulting from over weighting or under weighting single stocks in a sectors.

Interaction risk the residual risk, if present, coming from the interaction of the two previous effects

Currency risk independently from the previous risk types, the part of risk to be attributed solely on the dynamics of foreign-exchange rates

As already mentioned in the introduction, in this paper we will not consider the possibility of hierarchical trees but only single-level segments. The attribution analysis will be done only at one single level, either selecting a segment and a segment level, or choosing a general attribute on which the analysis is to be run.

4.2 Allocation risk

In portfolio attribution the allocation part of performance is assigned to the choice of sector weights. Similarly, the computation of allocation risk can be performed computing the extra risk given by the different risk exposure given to some portfolio sectors with respect to their benchmark counterparts.

In order to compute allocation risk we need to create a virtual portfolio in which each segment has the same outstanding value that has in the original portfolio, however with assets maintaining, inside each sector, the same relative weights as in the benchmark. In other words, the weights of the single assets are modified so that their relative weight inside the segment equal the corresponding relative weight in the benchmark, however, the collective weight of each segment is the same as that of the original portfolio.

Mathematically, consider a virtual portfolio, the allocation portfolio, so that for each asset \( i \) we define an outstanding value \( W^\text{allocat}_i \) given by,

\[
W^\text{allocat}_i = W^b_i \frac{W_a}{W^b_a},
\]

where the asset indexed by \( i \) is in a segment \( a \) (i.e. \( i \in a \)) and \( W^b_a \neq 0 \). Otherwise, i.e. for all assets that belong to a sector so that \( W^b_a = 0 \) we set,

\[
W^\text{allocat}_i = W_i.
\]

Figure 1 gives a graphical representation of the allocation-portfolio creation starting from the original portfolio and the benchmark. Note that for every sector in the universe, e.g. for sector \( a \), we have

\[
\sum_{i \in a} W^\text{allocat}_i = W_a,
\]

and similarly for the other sectors. We define allocation risk, written as \( R^\text{allocat} \), as the relative risk of the allocation portfolio with respect to the benchmark:

\[
R^\text{allocat} = R(P^\text{allocat} - B).
\]

In general allocation risk could be higher or lower than the active risk.

4.3 Decomposition of allocation risk

It is often of interest to compute how much allocation risk is contributed by each asset or sector. To this end we apply the relative-risk decomposition, as explained in section 3, to the relative allocation portfolio \( P^\text{allocat} - B \) and call \( C^\text{allocat}_i \) the contribution to the allocation risk of each asset so that

\[
R^\text{allocat} = R(P^\text{allocat} - B) = \sum_i C^\text{allocat}_i.
\]

We then define the contribution of segment \( a \) to allocation risk as,

\[
C^\text{allocat}_a = \sum_{i \in a} C^\text{allocat}_i.
\]
so that,
\[ C_{a}^{allocat} + C_{b}^{allocat} + C_{c}^{allocat} + \ldots = R^{allocat}. \]  (47)

Equations (45) and (47), respectively, are denoted as the \emph{allocation-risk decomposition} at asset level and at sector level.

### 4.4 Alternative: allocation risk at sector level

In the previous paragraph we decomposed allocation risk at asset and sector level. Alternatively, one could be interested in the allocation risk at sector level as described in the following. For each sector, for example sector \( a \), define the allocation portfolio \( P_{a}^{allocat} \) so that all assets that do not belong to sector \( a \) have the same weight as in the benchmark, however, assets in sector \( a \) have the same weight as in the allocation portfolio. Given the \( a \)-sector-based allocation portfolio we can define the \( a \)-sector allocation risk \( C_{a}^{allocat} \) as
\[ C_{a}^{allocat} = R(P_{a}^{allocat} - B). \]  (48)

Note that in general we have,
\[ C_{a}^{allocat} + C_{b}^{allocat} + C_{c}^{allocat} + \ldots \neq R^{allocat}. \]  (49)

even though we expect the difference, an interaction term between sectors, to be very small.

### 5 Selection risk

In performance attribution the selection performance, see for example reference [1], is the performance that can be attributed to stock picking: i.e. to the choice of over-exposing or under-exposing certain stocks in a sector. Stock picking also results in a higher or lower active risk. The fraction of active risk assigned to stock picking is called selection risk (less formally known as stock-picking risk). There are three different methods to compute selection risk:

- **Zero-interaction method.** Selection risk is the active risk not explained by allocation risk. (See subsection 5.1 for more details.)
- **Benchmark-weighted method.** A selection portfolio is built with sector weights as in the benchmark but relative weights inside a sector like in the portfolio. Selection risk is the risk of the selection portfolio relative to the original portfolio benchmark. (See subsection 5.2 for more details.)

- **Portfolio-weighted method.** Selection risk is the active risk of the actual portfolio with respect to a benchmark composed by the allocation portfolio. (See subsection 5.3 for more details.)

In the remainder of this section we analyze these three methods for computing selection risk and their consequences on the residual risk (interaction risk).

### 5.1 Zero-interaction selection risk

This is the simplest method to define selection risk: we measure the portfolio relative risk,

\[ R' = R(P - B), \]

and the allocation risk \( R_{\text{allocat}} \) defined by (44), and set the selection risk to be that part of relative risk not explained by allocation risk,

\[ R_{\text{select}} = R' - R_{\text{allocat}}. \]

Since we have \( R' = R_{\text{select}} + R_{\text{allocat}} \), we can say that allocation and selection risks exactly partition active risk.

Given the relative-risk decomposition at sector level, given by equation (38), and allocation risk at sector level, as written in equation (47), we define the decomposition of selection risk at sector level as

\[ C^\text{select}_a = C^r_a - R^\text{allocat}_a. \]

Similarly, given the asset-level relative-risk decomposition, see equation (34), and the asset level decomposition of selection risk, given by formula (45), we define

\[ C^\text{select}_i = C^r_i - R^\text{allocat}_i, \]

and interpret \( C^\text{select}_i \) as the component of selection risk associated to asset \( i \).

### 5.2 Benchmark-weighted selection risk

In order to compute the benchmark-weighted selection risk we need to create a virtual portfolio, the selection portfolio, composed by the same assets as the original portfolio, however, with segments having the same weights as they do in the benchmark. In order to achieve this result consider an asset \( i \) belonging to a segment \( a \) for which \( W_a \neq 0 \). Define the asset exposure \( W^\text{select}_i \) in the selection portfolio to be,

\[ W^\text{select}_i = W_i \frac{W^b_a}{W_a}. \]

On the other hand, when the segment outstanding value in the portfolio is null, i.e. \( W_a = 0 \), we need to create a segment with the same risk exposure as the corresponding segment in the benchmark, however, with the same weight it has in the portfolio, therefore we define,

\[ W^\text{select}_i = W_i + W^b_i, \]

for all assets in segment \( a \) where \( W_a = 0 \).

Given the selection portfolio \( P^\text{select} \), relative risk can be computed with respect to the benchmark and the result,

\[ R^\text{select} = R(P^\text{select} - B), \]

is called **benchmark-weighted** selection risk.

Note that computing the sum of the exposure of all assets in the same segment, e.g. segment \( a \), for the selection portfolio we obtain the benchmark exposure for that sector,

\[ \sum_{i \in a} W^\text{select}_i = W^b_a, \]

for both definitions (54) and (55). Hence, the portfolio stock picking is weighted on the benchmark sector weights inspiring the name **benchmark-weighted** for this computation of selection risk.
5.3 Portfolio-weighted selection risk

The third method to compute selection risk is based on the active risk between the original portfolio and the allocation portfolio. Stock picking is measured on sectors with the same risk exposure as they have in the portfolio.

In this method selection risk is computed as the active risk of the portfolio relative to a benchmark composed by the allocation portfolio:

\[ R^\text{select} = R(P - P^\text{allocat}). \] (58)

Since in this case stock picking is assumed to be performed once the portfolio allocation has already been decided, it is not unfair to say that this method mimics more closely the investment process.

5.4 Decomposition of selection risk

We have already shown in section 5.1 how to compute selection-risk components for the zero-interaction method at asset and sector level. In this section, using a similar method to that described for allocation risk, we derive a similar decomposition for the benchmark weighted and portfolio-weighted methods.

To obtain the contribution to selection risk at asset level, we can apply the risk decomposition technique described in section 3 to the active selection portfolio \( P^\text{select} - B \), for the benchmark-weighted method, and \( P - P^\text{allocat} \) for the portfolio-weighted method. Hence, we write \( C^\text{select}_i \) for the contribution to the appropriate active risk for each asset, so that,

\[ R^\text{select} = R(P^\text{select} - B) = \sum_i C^\text{select}_i, \] (59)

for the benchmark-weighted method and

\[ R^\text{select} = R(P - P^\text{allocat}) = \sum_i C^\text{select}_i, \] (60)

for the portfolio-weighted method. For both methods we define,

\[ C^\text{select}_a = \sum_{i \in a} C^\text{select}_i, \] (61)

as the contribution of segment \( a \) to selection risk. It is easy to prove that:

\[ R^\text{select} = C^\text{select}_a + C^\text{select}_b + C^\text{select}_c + \ldots, \] (62)
where the sum extends to all universe sectors.

**Alternative selection risk at sector level** Using a method similar to that described for allocation risk in sub-section 4.4, we could give an alternative definition of selection risk, both at asset and sector level, however, such detailed analysis is beyond the scope of the current paper.

### 5.5 Interaction risk

As shown in subsection 5.1, in the zero-interaction method relative risk is exactly the sum of allocation and selection risks. However, in general for the other two selection-risk methods, we will have:

\[ R' \neq R_{select} + R_{allocat}. \]  

(63)

We define interaction risk, and use the notation \( R^{int} \), as the residual difference between the portfolio risk and the sum of the selection and allocation risks:

\[ R^{int} = R' - R_{select} - R_{allocat}. \]  

(64)

Writing explicitly definition (64) in the benchmark-weighted method we have,

\[ R^{int} = R(P - B) - R(P_{select} - B) - R(P_{allocat} - B), \]  

(65)

while in the portfolio-weighted method we have,

\[ R^{int} = R(P - B) - R(P - P_{allocat}) - R(P_{allocat} - B). \]  

(66)

Rearranging terms in the last definition, for sub-additive risk measures we have

\[ R(P - P_{allocat}) + R(P_{allocat} - B) \geq R(P - B), \]  

(67)

from which we deduce

\[ R(P - B) - R^{int} \geq R(P - B), \]  

(68)

so that

\[ R^{int} \leq 0, \]  

(69)

i.e., interaction risk is always zero or negative.

Furthermore, comparing equation (65) and (66) we expect interaction risk to be smaller, in absolute value, when computed using the portfolio-weighted method than when computed using the benchmark-weighted method.

#### Decomposition of interaction risk

Since we have already computed the components of allocation risk and selection risks, we can also define a contribution to interaction risk at asset and sector level. For each asset \( i \) in the universe the interaction component is defined, analogously to definition (64), as the \( i \)-th relative-risk contribution minus the sum of selection and allocation components for that asset,

\[ C^{int}_i = C_i' - C_i^{select} - C_i^{allocat}. \]  

(70)

The interaction risk for a segment is defined, in the obvious way, as the sum of the interaction contributions from all assets in a segment, for example for segment \( a \),

\[ C^{int}_a = \sum_{i \in a} C^{int}_i = C'_a - C^{select}_a - C^{allocat}_a, \]  

(71)

or as the difference between the sector component of relative risk and the sum of the selection and allocation components for that sector. Finally, note that interaction risk components are always null when computed using the zero-interaction.

### 6 Currency risk

In order to increase performance and to diversify risk, often a portfolio manager buys products in different currencies. While such non-homogeneous portfolios could be, at least in principle, evaluated in any currency it is customary to choose a single evaluation currency: the *portfolio currency*.

In the computation of active risk in a portfolio with assets in different currencies the expected distribution is computed for each asset in the local currency and then adjusted, in each scenario, by the exchange-rate scenario (see, for example, reference [6]). In this way the computed risk correctly includes the dynamics of currency-exchange rates. For these portfolios it is fair to ask how much risk comes from the currency-exchange scenarios, the *currency risk*, and how much risk is intrinsic into the assets, the *local risk*.
In order to compute local risk for a given risk measure \( R \), we define a local-currency risk measure \( R^\text{loc} \) that returns risk exactly as the original measure, however, without including the currency scenarios. Hence, if \( R(P) \) denotes the computation of the full portfolio risk measure including the currency scenarios, the symbol \( R^\text{loc}(P) \) will denote the same computation without applying the currency scenarios. Therefore for any given portfolio \( P \) and an associated benchmark portfolio \( B \) we can define a currency risk measure \( R^\text{fx} \), as

\[
R^\text{fx} = R^\text{fx}(P - B) = R(P - B) - R^\text{loc}(P - B). \tag{72}
\]

Note that, for relatively well-behaved risk measures, all risk decomposition results derived in section 3 can be applied to the risk measures \( R^\text{loc} \) and \( R^\text{fx} \).

Similarly to section 4 we can define a local-currency allocation risk \( R^\text{loc-allocat} \) akin to the allocation risk defined in equation (44), with \( R^\text{loc} \) replacing \( R \),

\[
R^\text{loc-allocat} = R^\text{loc}(P \text{allocat} - B). \tag{73}
\]

Analogously, for each of the three methods described in section 5, we can define a local-currency selection risk \( R^\text{loc-select} \) using the local risk measure \( R^\text{loc} \) instead of \( R \). Hence, for the zero-interaction method we have

\[
R^\text{loc-select} = R^\text{loc} - R^\text{loc-allocat}, \tag{74}
\]

for the benchmark-weighted method we have,

\[
R^\text{loc-select} = R^\text{loc}(P \text{select} - B), \tag{75}
\]

and for the portfolio-weighted method

\[
R^\text{loc-select} = R^\text{loc}(P - P \text{allocat}). \tag{76}
\]

Finally we define a local interaction risk as

\[
R^\text{loc-int} = R^\text{loc}(P - B) - R^\text{loc-select} - R^\text{loc-allocat},
\]

for all selection-risk methods.

Given these definitions the relative portfolio risk can be written as,

\[
R^r = R^\text{fx} + R^\text{loc} = R^\text{fx} + R^\text{loc-select} + R^\text{loc-allocat} + R^\text{loc-int}. \tag{77}
\]

This decomposition of relative risk into currency risk, local selection risk, local allocation risk, and local interaction risk is sometimes called currency-based risk attribution.

### 6.1 Decomposition of risk attribution

As already mentioned earlier, the relative risk decomposition discussed in section 3 can also be applied to the risk computation of the \( R^\text{loc} \) risk measure. Hence, for each asset \( i \) we can compute \( C_i^\text{loc} \) so that,

\[
R^\text{loc} = R^\text{loc}(P - B) = \sum_i C_i^\text{loc}, \tag{78}
\]

and define the segment-level local contribution, for example for segment \( a \), as

\[
C_a^\text{loc} = \sum_{i \in a} C_i^\text{loc}. \tag{79}
\]

Similarly we can define the contribution to the currency risk

\[
C_i^\text{fx} = C_i^r - C_i^\text{loc}, \tag{80}
\]

for each asset \( i \) and for each segment, for example for segment \( a \), as

\[
C_a^\text{fx} = C_a^r - C_a^\text{loc}. \tag{81}
\]

Applying the relative-risk decomposition to the allocation portfolio using the local risk measure \( R^\text{loc} \) we can compute \( C_i^\text{loc-allocat} \), then using one of the three methods for the selection risk computation, we can define \( C_i^\text{loc-select} \). It is then straightforward to define the local interaction components at asset level as,

\[
C_i^\text{loc-int} = C_i^\text{loc} - C_i^\text{loc-select} - C_i^\text{loc-allocat}, \tag{82}
\]

and at sector level, e.g. for sector \( a \), as

\[
C_a^\text{loc-int} = C_a^\text{loc} - C_a^\text{loc-select} - C_a^\text{loc-allocat}. \tag{83}
\]

With all these definitions in mind we write the risk-attribution decomposition so that the asset-level relative-risk component \( C_i^r \) can be written as

\[
C_i^r = C_i^\text{fx} + C_i^\text{loc-select} + C_i^\text{loc-allocat} + C_i^\text{loc-int}, \tag{84}
\]
for every asset $i$ and for every segment, e.g. for segment $a$,

$$C_a = C_{a}^{fx} + C_{a}^{loc-select} + C_{a}^{loc-allocat} + C_{a}^{loc-int}.$$  (85)

In this way we obtained the currency-based decomposition of risk attribution.

### 7 Summary and conclusion

In this paper we analyzed the decomposition and the attribution of the active risk of a portfolio with respect to a benchmark. In section 3 we have shown how to compute relative risk and its decomposition first for tracking error, then for a generic simulation method and, finally, for a generic risk measure. In sections 4 and 5 we considered the allocation process more in details and split relative risk into an allocation risk, selection risk, and a residual interaction risk. Finally, in section 6, we considered portfolios that are also exposed to currencies different from the main portfolio currency and separated the currency part of risk from the remainder.

While in the current paper we focused on assets with a risk exposure equal to their outstanding value, i.e. assets with a future value written as in equation (4), the results obtained here are more general and can be applied to all types of assets such as futures or swaps (see, for example, reference [5]). Also, the risk decomposition and attribution developed in this paper is compatible with the risk-factor decomposition of reference [5], so that it is possible to decompose risk attribution into components associated to the portfolio risk factors.

### References


