Projection performance contributions of non-linear portfolios

Marco Marchioro
www.statpro.com
Version 0.97

January 2017*

Abstract

We define a theoretical framework to exactly measure the additive risk-driver contributions to the performance of an investment portfolio. The approach is based on first principles, is non local and is especially suitable for non-linear portfolios. We consider a single-period return and assume that the generated cash flows are reinvested in the portfolio itself. We find that the portfolio performance can be exactly split into a calendar component and a risk-driver contribution. For the risk-driver contribution we define the projection schema, i.e. we split it into the sum of terms rising from each risk driver separately and from the combination of multiple risk drivers (the cross terms). The results are obtained computing the portfolio return projections on the risk-driver axes and they are consistent with a full Taylor expansion of the approximating pricing function. Remarkably, the performance-contribution schema thus defined works well even for long evaluation periods and can be used in fixed-income attribution. Finally, we provide two practical examples: an equity option, where we find an important contribution for the compound equity/volatility component, and a portfolio of fixed-rate coupon bonds, for which we find that many cross contributions are identically zero.

Keywords: projection contributions, performance contributions, performance-contribution schema, fixed-income attribution, risk drivers, pricing functions, operator algebra, linear operators, projection operators.

Contents

*Latest review January 11, 2017

1 Introduction

The use of financial derivatives is no longer a novelty in asset management. Until a few decades we could have easily guessed which fund was extensively using derivatives and which one was not: equity funds were almost entirely invested in stocks, fixed-income funds in bonds, and only a handful of hedge funds would have a consistent part of their portfolio in derivatives. In recent years, financial innovation brought forth the use of derivatives to any type of funds, mostly to have an accurate control of the bets taken and to hedge unwanted risks. Hence we would have an hard time now to find a professionally-managed portfolio that does not contain any derivative at all. As a consequence, when we measure the performance of a modern portfolio, we can no longer ignore the presence of non-linear terms and we possibly need to consider multiple risk-driver classes at the same time (e.g., equities, interest rates, credit spread, and currency-exchange rates). Furthermore, because of the presence of hedging instruments it is often not possible to approximate the portfolio returns as a linear function of the risk-driver displacements. As a consequence there is a competitive advantage in being able to handle the portfolio non linearity.

Performance contributions In this world of non-linear assets the simple computation of a portfolio return tells us only part of the story (see, for example, reference [1] for a good review of performance-measurement techniques). In order to better understand their portfolio behaviour, asset managers are increasingly asking for analytics that can split the portfolio return into the sum of components, each of which has a clear financial meaning For example, for the simple case of an equity option we may want to write the return r over a past period as the sum of contributions coming from the equity itself, the equity volatility, and so on and so forth:

$$r = c^{\text{equity}} + c^{\text{volatility}} + \dots .$$

We call *performance contributions* the different components that add up to the portfolio return and *performance-contribution schema* the equation that provides the return in terms of the contributions. In this paper we define a performance-contribution schema that splits the portfolio performance into components that can be associated to one or more risk drivers.

The framework we describe relies on one major hypothesis: the possibility to compute the portfolio value as a function of the its intrinsic financial variables, namely the risk drivers. While this hypothesis might have not been satisfied in the past, nowadays, also thanks to the availability of open-source pricing libraries (see, for example, reference [4]), we are usually able to compute the portfolio value (at least approximately).

In this work we assume that a large number of players in the portfolio-management industry have the software and the financial data necessary to evaluate their portfolios. Therefore at any time t we can write the portfolio value $V(t)$ as a deterministic function,

$$V(t) = f(x_1, \dots, x_n; t) , \qquad (1)$$

of the *risk drivers* $x_1, \dots, x_n$ and the evaluation time t. In the above equation we suppose that all the portfolio variability comes from the risk drivers, i.e. the financial variables *steering* the portfolio value. We reserve the more popular term *risk factors* for the financial variables that influence the portfolio value in a stochastic fashion. We defer to reference [7] for a more detailed description of the difference between risk drivers and risk factors.

Perturbative contributions The standard approach to the computation of performance risk-driver contributions is perturbative, based on the Taylor expansion of the pricing function. For example, the performance of an equity-option premium evolving from f, observed at a time t, to f', observed at a later time t', can be computed as

$$\begin{aligned} f' - f &= c^t + c^\Delta + c^\Gamma \\ &\quad + c^r + c^\sigma + \varepsilon \\ &= -G_\Theta \cdot \Delta t + G_\Delta \cdot \Delta S + \tfrac{1}{2} G_\Gamma \cdot (\Delta S)^2 \\ &\quad + G_\rho \cdot \Delta r + G_\upsilon \cdot \Delta\sigma + \varepsilon\,, \end{aligned} \tag{2}$$

where $\Delta t = t' - t$, G_Θ is the option time-to-maturity derivative (a.k.a. the Greek *theta*), G_ρ is the option interest-rate derivative (a.k.a. the Greek *rho*), G_Δ is the option equity first derivative (a.k.a. the Greek *delta*), G_Γ is the option equity second derivative (a.k.a. the Greek *gamma*), G_υ is the option volatility derivative (a.k.a. the Greek *vega*). Also, in the above expression, Δr, ΔS, and $\Delta\sigma$ are, respectively, the interest-rate, the equity, and the volatility displacements. Finally the residual ε, not necessarily small, has been introduced to exactly recover the price variation $f' - f$.

Similarly in the fixed-income world, the return on a fixed-rate coupon bond is typically split as (see, for example, reference [5])

$$\begin{aligned} \frac{f' - f}{f} &= c^t + c^{\text{durat}} + c^{\text{conv}} + \varepsilon \\ &= y \cdot \Delta t - M_D \cdot \Delta y + \tfrac{1}{2} C_V\,(\Delta y)^2 + \varepsilon\,, \end{aligned} \tag{3}$$

where y is the bond yield, M_D is the modified duration, C_V is the bond convexity (for precise definitions of M_D and C_V see reference [3]), and the residual ε was again introduced to account for other unknown terms.

The perturbative techniques to compute the risk-driver contributions work reasonably well in many cases, however, they have few major shortcomings:

1. they needs to be specifically engineered for each pricing function,
2. there are no guarantees that the residual term ε is small,
3. and, in the case of hedged portfolios, they typically results in very large residuals.

The first limitation becomes a problem, especially for portfolios with many different types of instruments, as the contribution form one asset may not be comparable with that from another asset. This problem, however, it is easily solved. For example, we describe in appendix A a generic approach to the computation of the second-order perturbative schema. The second weak point is more serious: if any contribution is smaller than the residual, it means that the residual tells us more, i.e. contains more information, about the performance than some computable contributions. Dismissing the residual as some kind of interaction only sweeps the problem under the carpet. The third and final shortcoming is more apparent for delta-hedged portfolios, or even worse for gamma-hedged portfolios. For these portfolios most sensitivities are close to zero by construction, so that a perturbative approach is bound to fail. In this paper we propose an improved solution to the above problems that results into a null residual using a finite number of terms.

In equations (2) and (3) we focus on the risk-driver displacements, instead, the priority should be given to the portfolio-value variations caused by the risk-driver displacements. Shifting the focus from the displacements of the risk drivers to that of the portfolio value, and by carefully estimating all performance contributions, even those that appears to be very small at first, we reach our goal of creating a precise portfolio-contribution schema.

Hypothesis and goals Remarkably the hypothesis needed in building the contribution framework are rather weak, resulting in the coverage of a wide range of portfolios under a variety of market conditions. We compute the performance in a single period, also referred in the paper as *the period*. We require that all the generated cash flows are reinvested in the portfolio itself, i.e. that there are no exchanges of cash going in or out of the portfolio (an hypothesis that can be easily relaxed by keeping track of the cash flows and by categorizing them into a separate contributions). Finally, we assume that we have a known approximating pricing function for the portfolio value at the beginning and at the end of the period and that the corresponding risk drivers are known.

It is also important to note the assumptions we *do not* make, notably that the observed period is small, or that the pricing function is smooth (even thought these assumption are usually satisfied).

Again, the goal of this paper is to split the single-period portfolio performance into a number of additive contributions each of which can be identified to arise from either a single risk driver or from the compounding effect of multiple risk drivers.

2 Performance contributions for simple portfolios

Before going into the details of the projection-contribution schema, in this section, we consider the example of portfolios that can be modelled using simple pricing functions. Specifically we consider what may well be the easiest example of non-trivial computation of performance contributions: the contributions to the performance of a stock in a foreign currency. This example is also useful to familiarize with the concept of compound effect of two risk drivers. We then generalize the results to portfolios that can be modelled using a pricing function proportional to the mathematical product of its risk drivers. This second type of pricing function will allow us to familiarize with the compounding effects of three or more risk drivers.

2.1 Performance contributions for an equity in a foreign currency

We consider the case of a portfolio with a single stock in a foreign currency as a starting point to analyse performance contributions. However, in order to compute the currency and the equity performance contributions, we need to understand first the case of an amount of cash held in a foreign currency and that of a stock in its own currency.

Return of cash held in a foreign currency Consider an investor with a portfolio in Euro with a value, at the beginning of the period, of $V_s = 80$€, investing the whole amount in a foreign currency, e.g. in 100\$. Ignoring interest-rate differentials, suppose that at the end of the period the 100\$ dollars could be exchanged back for $V_e = 82$€ (note that, in order to compute the return, the transaction doesn't necessarily need to occur). We define the *wealth ratio* X as

$$X = \frac{V_e}{V_s} = \frac{82\,€}{80\,€} = 1.025\,,$$

the portfolio *performance* can then be computed as

$$r^x = X - 1 = 2.5\,\%\,. \tag{4}$$

All the performance contributions to this portfolio come from variations in the foreign-exchange rates, hence, it is fair to state that r^x has only one contribution: the currency component.

Return of a single stock Suppose now that another investor has a tiny portfolio composed by a single stock, paying no dividends, in the local currency: for example an equity traded in US dollars on the NYSE. If the stock, at the beginning of the period, is worth V_s=100\$ and its value, at the end of the period, is V_e=110\$ we can compute the wealth ratio Y as,

$$Y = \frac{V_e}{V_s} = \frac{110\$}{100\$} = 1.1\,.$$

Then the stock return over the period can be computed as

$$r^y = Y - 1 = 10\%\,. \tag{5}$$

Since we are not considering any other risk driver the equity contribution r^y is the only performance component to this portfolio.

Compound return of a stock in a foreign currency Consider now a third investor with a portfolio in the Euro currency that, over the same time period, exchanges its 80€ to 100\$ and then invests those money into the same stock as the second investor. The portfolio value at the beginning of the period is still given by V_s=80€, however at the end of the period it turns out to be

$$V_e = \frac{110}{100} 82\,€ = 90.2\,€\,,$$

because this investor could exchange his stock for 110$ and then exchange those money, at the same exchange-rate as the first investor, to obtain 90.2€(again, in order to compute the return we do not actually need to perform the transaction, but only evaluate the potential portfolio value). The composite wealth ratio W is then given by the product of the equity and the currency-exchange wealth ratios,

$$W = \frac{V_e}{V_s} = X \cdot Y = 1.1275\,,$$

and the compound return of the stock and the currency-exchange rate can be computed as

$$r^w = W - 1 = X \cdot Y - 1 = 12.75\,\%\,. \qquad (6)$$

The question now is: how much of this return comes from the equity and how much does it come from the foreign currency? To answer this question we go back to the return of the first investor and his performance for holding the foreign currency: r^x=2.5 %, independently from the performance of any stock. Then we consider the second investor with a simple return of $r^y = 10\,\%$, in this case independently from any exchange rate. Finally we examine the compound return of the third investor in Euro, i.e. $r^w = 12.75\,\%$ depending both on the equity performance and the exchange-rate performance. Note that the sum of the stock return and the foreign-currency return does not equal the compound return,

$$\begin{aligned} r^x + r^y &= 2.5\,\% + 10\,\% = 12.5\,\% \\ &\neq \;\; 12.75\,\% = r^c\,. \end{aligned}$$

and that the difference,

$$r^{x*y} = r^w - (r^x + r^y) = (X-1)\cdot(Y-1) = 0.25\,\%\,, \qquad (7)$$

can only be attributed to the compounding effect of the equity and the exchange-rate risk drivers (see reference [1] for a more intuitive graphical explanation of this compounding effect).

In summary, we can *exactly* split the performance of a stock in a foreign currency as the sum of three components:

$$r^c = r^x + r^y + r^{x*y}\,, \qquad (8)$$

a contribution coming from each risk driver, i.e. the equity spot price and the foreign-exchange rate, and a *cross contribution* coming from the compounding of the two risk drivers (however excluding the simple returns due to each risk driver individually).

The goal of this paper is to extend the analysis just carried to the performance of a generic portfolio depending on a any number of risk drivers. We can expect the portfolio performance to have contributions from each single risk driver and cross contributions arising from the compounding of multiple risk drivers.

2.2 Portfolios depending on the product of several risk drivers

In the previous subsection we explicitly separated the return generated by a stock in a foreign currency into three risk-driver contributions. In this subsection we prove some algebraic expressions that are useful to generalize the concept of performance contributions to the generic case of a portfolio modelled by a pricing function depending on several risk drivers. For the time being we make the very-restrictive assumption that the portfolio pricing function is multiplicative in all risk drivers and we show in later sections that this results can be modified to allow for generic pricing functions.

Multiplicative pricing functions depending on two variables The result of subsection 2.1 can be formally written as a simple algebraic equation considering the multiplicative pricing function $f(X,Y) = X \cdot Y$, in two variables X and Y. Note that the compound return is given by

$$m_2(X,Y) = X \cdot Y - 1\,,$$

the notation m_2 was used to stress the dependence on two variables. Initially the two variables X and Y can be thought to be the wealth ratios needed to compute the compound return, however the reader should keep in mind that the following algebraic manipulations allow X and Y to be any reasonable mathematical objects (in section 3 they will actually be linear operators). More in general we are interested in splitting $m_2(X,Y)$ using the two binomials $(X-1)$ and $(Y-1)$. To this end we notice the following simple algebraic equation

$$X \cdot Y - 1 = (X-1) + (Y-1) + (X-1)\cdot(Y-1)\,, \quad (9)$$

where the parenthesis have been added for clarity. Hence we can write

$$m_2(X,Y) = C^X + C^Y + C^{X*Y}\,. \quad (10)$$

with the following definitions for the X and Y contributions:

$$C^X = X - 1\,, \qquad \text{and} \qquad C^Y = Y - 1\,. \quad (11)$$

Similarly we define the cross return C^{X*Y} as

$$C^{X*Y} = (X-1)\cdot(Y-1)\,, \quad (12)$$

characterizing the effect on performance given by the co-movements of both risk drivers.

Observe also that the cross return C^{X*Y} can be geometrically interpreted to be the area of the rectangle with sides given by the performances C^X and C^Y.

Multiplicative pricing function depending on three variables We consider the performance contributions for a multiplicative pricing function of three variables:

$$f(X,Y,Z) = X \cdot Y \cdot Z\,,$$

so that the portfolio return can be written as,

$$m_3(X,Y,Z) = X \cdot Y \cdot Z - 1\,.$$

It is easy to show the following algebraic equality:

$$\begin{aligned} X \cdot Y \cdot Z - 1 = {} & (X-1) + (Y-1) + (Z-1) \\ & + (X-1)\cdot(Y-1) + (X-1)\cdot(Z-1) \\ & + (Y-1)\cdot(Z-1) \\ & + (X-1)\cdot(Y-1)\cdot(Z-1)\,, \end{aligned}$$

so that the return m_3 can be split as

$$\begin{aligned} X \cdot Y \cdot Z - 1 = {} & C^X + C^Y + C^Z \\ & + C^{X*Y} + C^{X*Z} + C^{Y*Z} \\ & + C^{X*Y*Z}\,, \end{aligned} \quad (13)$$

i.e. as the sum of contributions C^X, C^Y, and C^Z, coming from the single variables, as defined by equation (11), from the pairwise contributions C^{X*Y}, C^{X*Z}, and C^{Y*Z}, defined similarly to equation (12), and finally from the *triplet contribution*,

$$C^{X*Y*Z} = (X-1)\cdot(Y-1)\cdot(Z-1)\,. \quad (14)$$

The triplet return contribution C^{X*Y*Z} derives from the compounding effects of the three risk drivers X, Y, and Z, explained neither by the single risk-driver returns nor by the pairwise cross returns coming from each risk-driver pair. Geometrically the triplet performance contribution C^{X*Y*Z}, given by equation (14), can be thought to be the volume of a rectangular cuboid with sides given by C^X, C^Y, and C^Z.

At first sight equation (13) does not seem to have any applicability to actual financial portfolios, however as we are going to show in subsection 3.4, with the appropriate definitions of X, Y, and Z, this equation can indeed be related to the risk-driver contributions of the performance of a pricing function depending on three risk drivers (see also subsection 4.1 for an explicit computation of these contributions).

Multiplicative pricing function depending on several variables We generalize the results obtained so far for two and three variables to a multiplicative pricing function depending on a number n of variables $X_1, X_2, \ldots, X_n$. We are going to show that the performance of a multiplicative pricing function of several variables can be written as

$$C^f = C^{\text{drivers}} + C^{\text{pairs}} + C^{\text{triplets}} + \ldots + C^{n\text{-plet}}\,, \quad (15)$$

where C^{drivers} is the sum of all returns coming from the single risk drivers, e.g. $X_1 - 1$; C^{pairs} is the sum of all cross returns coming from all risk-driver pairs, e.g. $(X_1 - 1)\cdot(X_2 - 1)$; C^{triplets} is the sum of the triplet returns coming from three risk drivers such as $(X_1-1)\cdot(X_2-1)\cdot(X_3-1)$, and so on and so forth for all other orders; finally, $C^{n\text{-plet}} = (X_1-1)\cdot\ldots\cdot(X_n-1)$ is the (single) n-plet contribution coming from the compounding of all risk drivers. We provide in the remainder of this section the proof of equation (15), the reader not interested in these mathematical details can safely skip this part and continue to read from subsection 2.3.

For the formal proof of equation (15), consider the portfolio return $m_n(X_1,\ldots,X_n)$ defined as

$$m_n(X_1,X_2,\ldots,X_n) = X_1\cdot X_2\cdot\ldots\cdot\ X_n - 1\,. \quad (16)$$

We write equation (15) as

$$\begin{aligned} m_n(X_1,\ldots,X_n) = & \quad (17)\\ & \sum_{i=1}^{n} C^{X_i} + \sum_{\{i,j\}\in\mathcal{C}_2^n} C^{X_i*X_j} + \ldots\\ & \ldots + \sum_{\{j_1,j_2,\ldots j_k\}\in\mathcal{C}_k^n} C^{X_{j_1}*X_{j_2}*\ldots*X_{j_k}}\\ & \ldots + C^{X_1*X_2*\ldots*X_{n-1}*X_n}\,, \end{aligned}$$

where $\mathcal{C}_2^n$ denotes the set of $\binom{n}{2}$ distinct pairs taken from the n indexes $j_1,j_2,\ldots j_n$, $\mathcal{C}_3^n$ is the set of $\binom{n}{3}$ distinct triplets of indexes, and so on and so forth. With the generic notation $\mathcal{C}_k^n$ we denote the set of $\binom{n}{k}$ distinct k-plet of n indexes, so that

$$C^{X_{j_1}*X_{j_2}*\ldots*X_{j_k}} = (X_{j_1}-1)\cdot(X_{j_2}-1)\cdot\ldots\cdot(X_{j_k}-1)\,, \quad (18)$$

is defined for any set of indexes $\{j_1,j_2,\ldots j_k\}$ belonging to $\mathcal{C}_k^n$.

Equation (17) can be obtained by mathematical induction on the number of variables n. For n=1 the result is trivial since

$$m_1(X_1) = X_1 - 1 = C^{X_1}\,.$$

For n=2 equation (17) amounts to expression (10) and for n=3 it reduces to formula (13).

By induction, let's assume that equation (17) is true for $n-1$, i.e. that

$$\begin{aligned} m_{n-1}(X_1,\ldots,X_{n-1}) = & \sum_{i=1}^{n-1} C^{X_i} + \quad (19)\\ & \sum_{\{i,j\}\in\mathcal{C}_2^{n-1}} C^{X_i*X_j} +\\ & \ldots + \sum_{\{i,j,k\}\in\mathcal{C}_3^{n-1}} C^{X_i*X_j*X_k} +\\ & \ldots + C^{X_1*X_2*\ldots*X_{n-1}}\,. \end{aligned}$$

Then, for $n>1$, we consider the following algebraic equality

$$\begin{aligned} m_n(X_1,\ldots,X_n) &= X_1\cdot\ldots\cdot\ X_n - 1\\ &= (m_{n-1}+1)\cdot(X_n-1+1)-1\\ &= m_{n-1}\cdot(X_n-1)\\ &\qquad + (X_n-1) + m_{n-1}\,, \end{aligned}$$

and compare this expression with equation (17). Expression m_{n-1} (i.e. the last term above) has all the terms of equation (17) that do not contain X_n-1. We then have the contribution X_n-1 given by the n-th risk-driver alone. Finally the expression $m_{n-1}\cdot(X_n-1)$ contains all the contributions of equation (17) obtain by the compounding of X_n-1 with all the other risk driver, thus completing our proof.

Note that m_n can be synthetically written as,

$$\begin{aligned} m_n(X_1,\ldots,X_n) = & \sum_{i=1}^{n} C^{X_i} \quad (20)\\ & + \sum_{k=2}^{n} \sum_{\{j_1,\ldots j_k\}\in\mathcal{C}_k^n} C^{X_{j_1}*X_{j_2}*\ldots*X_{j_k}}\,, \end{aligned}$$

i.e. the sum of all the n contributions coming from each single risk driver X_i, the basic contributions, and the 2^n-n-1 cross contributions coming from each pair of variables, each triplet of variables, and so on.

Once again we note that the cross contributions $C^{X_{j_1}*X_{j_2}*\ldots*X_{j_k}}$'s account for the extra compounding of all the risk drivers X_{j_1}, ..., X_{j_k}'s not explained by the risk drivers themselves, their pairs, their triplets and so on and so forth. Finally, similarly to earlier remarks, the k-driver performance contribution $C^{X_{j_1} * X_{j_2} * \ldots * X_{j_k}}$, given by definition (18), can be thought to be the hyper-volume of the hyper-cube with sides $C^{X_{j_1}}, \ldots, C^{X_{j_k}}$.

2.3 Generic risk drivers and pricing functions

The model pricing functions considered so far are just examples useful to understand the effect of compounding when more than one risk driver is involved. When considering actively-managed portfolios we need to be able to contemplate a much wider class of pricing functions. Hence, in this subsection we consider portfolios with a value that can be modelled by a generic pricing function, depending on any number of risk drivers. Note that portfolio under consideration may not be the traded portfolio and it could be a sub-portfolio useful to the performance-management team. For example in a foreign fixed-income portfolio, where two different teams manage the currency-exchange rates and the bond trading, we might choose to focus on the bond portfolio in the local currency. Another example is attribution of a portfolio against a benchmark in which we would focus on the active portfolio.

The monetary value of a generic portfolio can often be modelled as depending on a number n of risk drivers. We assume the risk drivers to be observable on the market and that we are able to evaluate the portfolio at any time by the knowledge of their quotes.

Examples of pricing functions The value of the simple portfolio containing a single stock in a foreign currency, as seen in section 2.1, can be mathematically modelled as a function of $n=2$ risk drivers,

$$f(x, s) = x \cdot s \,, \tag{21}$$

the equity spot price s and the currency-exchange rate x.

Another example is the value of an equity-only portfolio invested in k stocks, each in a different currency, which, neglecting the effects of liquidity risk, can be represented by the sum of the single equity positions:

$$f(s_1, x_1, \ldots, s_k, x_k) = q_1 \cdot s_1 \cdot x_1 + \ldots + q_n \cdot s_k \cdot x_k \,, \tag{22}$$

where q_i, s_i, and x_i, for $i = 1, \ldots, k$, are, respectively, the quantity, the spot price, and the exchange rate of the i-th equity. Note how the pricing function defined in equation (22) depends on $n = 2 \cdot k$ risk drivers.

In the fixed-income world we can consider, for example, the simple portfolio composed by a single fixed-rate coupon bond paying n coupons $a_1, \ldots, a_n$, at future maturities $T_1, \ldots, T_n$. Assuming the redemption to be included in the last coupon a_n, we can write the portfolio value as

$$f(y_1, \ldots, y_n; t) = \frac{a_1}{(1 + y_1)^{(T_1 - t)}} + \ldots \tag{23}$$
$$\ldots + \frac{a_n}{(1 + y_n)^{(T_n - t)}} \,,$$

where, in this case, the n risk drivers are the term-structure yields $y_1, \ldots, y_n$.

In the equity-derivative world one often needs to include the equity volatilities in the risk-driver list. For example, we can write the pricing function of a very simple portfolio composed only by a long plain/vanilla equity option on a non-dividend-paying stock as

$$f^{\text{call}}(s, r, \sigma; t) = BS^{\text{call}}(s, r, \sigma; T - t) \,, \tag{24}$$

where BS^{call} represents the Black-Scholes-Merton pricing formula, $T - t$ is the time to maturity, and the risk drivers s, r, σ, are respectively the underlying-stock price, the appropriate risk-free interest rate, and the implied option volatility. In this example one should not include the strike as a risk driver since its value it's only a fixed parameter and does not vary from one day to the next.

We have described so far only a few examples from the universe of possible pricing functions that can be found in the real world. As we have remarked in section 1, most portfolios include derivative assets in order to enhance performance, or decrease risk, hence giving rise to very complex pricing functions.

2.4 Portfolio cash performance

In general the pricing function depends on the risk drivers as well as the evaluation time. We write the portfolio initial value, i.e. the value at the beginning of the period t, as

$$V_s = f(x_1, \ldots, x_n; t) ,$$

where $x_1, \ldots, x_n$ denote the risk drivers at time t, and its final value, i.e. the value at the end of the period t', as

$$V_e = f(x'_1, \ldots, x'_n; t') ,$$

with the risk driver values at t' denoted by $x'_1, \ldots, x'_n$. Assuming that there are no cash flows in or out of the portfolio the performance can be computed as (see reference [1] for more details)

$$r = \frac{V_e}{V_s} - 1 = \frac{V_e - V_s}{V_s} = \frac{\Delta f}{V_s} ,$$

where we define the *portfolio cash performance* Δf as

$$\Delta f = f(x'_1, \ldots, x'_n; t') - f(x_1, \ldots, x_n; t) . \qquad (25)$$

In the remainder of this paper we focus on the portfolio cash performance Δf and consider its split in additive risk-driver contributions.

Once again, the aim of this work is to split the portfolio cash performance into a number k of contributions:

$$\Delta f = c^1 + c^2 + \ldots + c^k ,$$

so that we can write the portfolio return as

$$r = r^1 + r^2 + \ldots + r^k , \qquad (26)$$

with

$$r^i = \frac{c^i}{V_s} \qquad \text{for all } i . \qquad (27)$$

In terms of notation, since in the remainder of the paper we mostly consider the portfolio cash performance and not the relative return, for brevity, usually we drop the word *cash* from "portfolio cash performance."

Performance contribution schemata As shown in reference [7], any procedure that allows us to split the ex-post performance into additive components, as in equation (26), is called an *(ex-post) performance-contribution schema*. Performance-contribution schemata are important because they allow the portfolio managers to have a specific *view* on the origin of the performance in their portfolios.

For example, we could define a performance-contribution schema that splits the portfolio returns into geographical components or into industry-sector contributions. In performance attribution (see, again, reference [1]), one often writes the portfolio active return as

$$r = r^{\text{alloc}} + r^{\text{select}} + r^{\text{currency}} ,$$

i.e. the sum of an allocation contribution, a selection component, and a currency term. The use of different contribution schemata for the same portfolio (or active portfolio) provide a better understanding of the portfolio performance.

In the next section we define the *projection contribution schema* that classifies the return contributions according to the portfolio risk drivers.

3 The projection contribution schema

As we have seen at the end of the last section, the portfolio manager may focus on different performance-contribution schemata to better understand its portfolio. This section is the heart of the paper: we define the projection-contribution schema and split the performance into a calendar component, a number of terms coming from each risk driver, and the contributions arising from the compounding effects of multiple risk drivers.

3.1 Calendar performance component

The first contribution we can isolate in the portfolio performance is the component originating merely from the effect of passing time. The question we ask is: *what would the portfolio performance be if all risk drivers were constant during the period?* For any time-independent pricing function the obvious answer would be zero, however, for an explicitly time-dependent pricing function we can identify the *calendar component* with

$$c^t = f(x_1, \dots, x_n; t') - f(x_1, \dots, x_n; t) \,. \qquad (28)$$

This term divided by the portfolio value at the start of the period, namely $r^t = c^t / V_s$, is the *calendar-return* contribution first introduced by reference [6]. Sometimes in the environment of fixed-income attribution this component is also known as the *carry* term.

Given the above definition of the calendar component we split the cash performance, defined in equation (25), into two parts:

$$\Delta f = c^t + c^d \,, \qquad (29)$$

with the *risk-driver contribution* to performance c^d given by

$$c^d = f(x'_1, \dots, x'_n; t') - f(x_1, \dots, x_n; t') \,. \qquad (30)$$

By definition the calendar component c^t is the part of performance identifiable with the ordinary passage of time. The risk-driver contribution c^d, on the other hand, answers another question: as seen from the end of the period t', *what is the performance contribution that can be attributed to the sole evolution of the underlying risk drivers?* Note that, while the contribution c^t is relatively simple to compute, the term c^d generally has a very rich structure and needs to be further split into sub-components.

While in the remainder of this subsection we focus on the exact computation of the calendar component, we refer to appendix A for the traditional perturbative calculation.

Exact computation of time component When the portfolio pricing function is known, as we always assume in this paper, we should favor the direct use of equation (28) for the computation of time component. In the case of a fixed-rate coupon bond with a pricing function given by equation (23), for example, the time contribution can be computed as

$$\begin{aligned} c^t \;=\; & \frac{a_1}{(1+y_1)^{(T_1-t')}} - \frac{a_1}{(1+y_1)^{(T_1-t)}} + \dots \\ & \dots + \frac{a_n}{(1+y_n)^{(T_n-t')}} - \frac{a_n}{(1+y_n)^{(T_n-t)}} \,, \end{aligned}$$

which yields,

$$c^t \;=\; a_1 \frac{(1+y_1)^{\Delta t} - 1}{(1+y_1)^{(T_1-t)}} + \dots + a_n \frac{(1+y_1)^{\Delta t} - 1}{(1+y_n)^{(T_n-t)}} \,.$$

Continuously compounded bond yield Alternatively to equation (23), we can use a single yield risk driver y to express the continuously-compounded discount factor as e^{-yt}. In this case we define the bond pricing function as

$$f(y;t) = a_1 \cdot e^{-y(T_1-t)} + \dots + a_n \cdot e^{-y(T_n-t)} \,, \qquad (31)$$

so that we can compute the carry contribution as

$$\begin{aligned} c^t \;=\; & a_1 \left[e^{-y(T_1-t')} - e^{-y(T_1-t)} \right] + \dots \\ & \dots + a_n \left[e^{-y(T_n-t')} - e^{-y(T_n-t)} \right] \,, \end{aligned}$$

that can be simplified to give

$$\begin{aligned} c^t \;=\; & a_1 \cdot e^{-y(T_1-t)} \left[e^{y\cdot\Delta t} - 1 \right] + \dots \\ & \dots + a_n \cdot e^{-y(T_n-t)} \left[e^{y\cdot\Delta t} - 1 \right] \\ =\; & \left[e^{y\cdot\Delta t} - 1 \right] f(y;t) \,. \qquad (32) \end{aligned}$$

Using this expression and equation (27) we can compute the bond calendar return component as

$$r^t = \frac{c^t}{f(y;t)} = e^{y\cdot\Delta t} - 1 \,, \qquad (33)$$

that, in the limit $\Delta t \to 0$, results in the traditional carry term computation

$$r^t \simeq y \cdot \Delta t \,. \qquad (34)$$

While the traditional fixed-income carry effect, as computed in the simplified form (34), is valid only for small time displacement Δt, equation (33) provides more general expression that can be used for arbitrarily-large periods.

Accrual and convergence contributions In fixed-income attribution the carry term c^t is often split into an accrual term and a convergence term. The accrual term results from the presence of coupons at a known interest rate, hence, does not depend explicitly on any risk driver. In order to compute this term we split the pricing function into a coupon-dependent term $f^a(t)$ (however independent on any risk driver) and a clean price:

$$f(x_1, \ldots, x_n; t) = f^a(t) + f^{\text{clean}}(x_1, \ldots, x_n; t). \quad (35)$$

note that the function $f^a(t)$ is typically linear, or piece-wise linear, in the time variable t.

Using equation (35) and the calendar-component definition (28) we separate c^t into accrual and convergence components:

$$c^t = c^t_{\text{acc}} + c^t_{\text{conv}}, \quad (36)$$

with

$$c^t_{\text{acc}} = f^a(t') - f^a(t), \quad (37)$$

and

$$c^t_{\text{conv}} = f^{\text{clean}}(x_1, \ldots, x_n; t') - f^{\text{clean}}(x_1, \ldots, x_n; t). \quad (38)$$

The term c^t_{acc} summarises the deterministic coupon return across the observation period. On the other hand, the term c^t_{conv} can be thought as the time contribution not due to accruals and purely related to the passage of time. For example we might find the c^t_{conv} term significant in a portfolio with many zero-coupon bonds, where the pull-to-par effect is important. The convergence term could also be substantial in a portfolio with plain/vanilla options where the time-decay effect is relevant, especially for those options close to the expiration date.

As an explicit fixed-income example, consider the fixed-rate bond with a pricing function described by equation (23). We can write

$$f(y_1, \ldots, y_n; t) = f^{\text{clean}}(y_1, \ldots, y_n; t) + f^a(t), \quad (39)$$

with

$$f^a(t) = a_1 \frac{t - T_0}{T_1 - T_0}, \quad (40)$$

where T_0 and T_1 denote the coupon accrual beginning and end dates. The accrual contribution can then be readily computed as

$$\begin{aligned} c^t_{\text{acc}} &= f^a(t') - f^a(t) = a_1 \frac{t' - t}{T_1 - T_0} \\ &= a_1 \frac{\Delta t}{T_1 - T_0}, \end{aligned} \quad (41)$$

which is exactly the accrued coupon from t to t'.

3.2 Risk-driver contribution

Earlier in this section we split the portfolio performance into a calendar component and a risk-driver contribution and show how to compute exactly the former in some special cases.

Risk-driver contributions alternative computation Using the split of the time contribution from equation (36), we can write the portfolio model performance of equation (29) as

$$\Delta f = c^t_{\text{acc}} + c^t_{\text{conv}} + c^d. \quad (42)$$

At this point we can slightly simplify the computation of the risk-driver contribution c^d, see definition (30), by extracting from the pricing function the accrual part, see equation (35), to obtain

$$c^d = f^{\text{clean}}(x'_1, \ldots, x'_n; t') - f^{\text{clean}}(x_1, \ldots, x_n; t'). \quad (43)$$

There are practical cases in which we might prefer this equation, instead of expression (30), to compute the risk-driver contribution.

Simplified notation In terms of notation, since the pricing function is only being evaluated at the end of the period t', when there is no possibility of confusion we write

$$c^d = f(x'_1, \ldots, x'_n) - f(x_1, \ldots, x_n). \quad (44)$$

Using this notation the following formulas look slightly less cumbersome. Furthermore, it is useful to write the risk drivers at t' as displacements with respect to the risk-driver value at the beginning of the period:

$$x'_1 = x_1 + h_1 \qquad \ldots \qquad x'_n = x_n + h_n \,. \tag{45}$$

We call the quantities $h_1, \ldots, h_n$, the *risk-driver displacements*.

3.3 Exact contributions from two risk drivers

In appendix A we show how to use the perturbative approach to compute the risk-driver contributions. The major drawback of the perturbative approach is the presence of a performance residual that cannot be explained. We consider here the definition of a performance-contribution schema that doesn't give rise to any residual term. Note that, while in the perturbative approach we assume all h_i's to be small, in the approach that follows we do not make this assumption.

Firstly we focus on the risk-driver contribution c^d and its computation for the special case when only two risk drivers are present in the pricing function (i.e. for n=2). We explain the method by initially assuming the pricing function to be infinitely differentiable, however, we will see later that this assumption can be relaxed.

The complete Taylor-series representation We consider a portfolio that can be evaluated using a pricing function $f(x, y)$ depending on two risk drivers x and y. Using the full Taylor representation of $f(x + h_x, y + h_y)$ we can write the portfolio (cash) performance as

$$\begin{aligned} c^d &= f(x + h_x, y + h_y) - f(x, y) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{h_x^{n-k}}{(n-k)!} \frac{h_y^k}{k!} \frac{\partial^{n-k}}{\partial x^{n-k}} \frac{\partial^k}{\partial y^k} f \,, \end{aligned} \tag{46}$$

where we used the notation that

$$\frac{\partial^0}{\partial x^0} = \frac{\partial^0}{\partial y^0} = \mathbf{1}$$

represent the identity operator, defined as

$$\mathbf{1}\, f = f \,. \tag{47}$$

Note that equation (46) uses the *full Taylor representation* so that the pricing function is recovered exactly. This is in contrast with the more common approach of using the *approximate Taylor expansion*, as describe, for example, in appendix A.

We can round up, in equation (46), the terms depending only on first risk-driver displacement h_x and call it c^x, then we collect all terms depending only on h_y and call them c^y; finally gather the leftovers, that at this point must depend on both h_x and h_y, to create the term c^{x*y}. In other words we write the risk-driver component as

$$c^d = c^x + c^y + c^{x*y} \,, \tag{48}$$

More explicitly the term c^x is computed adding all terms of equation (46) with $k = 0$:

$$\begin{aligned} c^x &= \sum_{n=1}^{\infty} \frac{h_x^n}{n!} \frac{\partial^n}{\partial x^n} f \\ &= h_x \frac{\partial f}{\partial x} + \frac{h_x^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{h_x^3}{3!} \frac{\partial^3 f}{\partial x^3} + \ldots \,. \end{aligned} \tag{49}$$

We use a similar definition for c^y by taking all terms with $k = n$. Then we define the term c^{x*y} as the leftover terms:

$$\begin{aligned} c^{x*y} &= \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \frac{h_x^{n-k}}{(n-k)!} \frac{h_y^k}{k!} \frac{\partial^{n-k}}{\partial x^{n-k}} \frac{\partial^k}{\partial y^k} f \\ &= h_x\, h_y \frac{\partial^2 f}{\partial x\, \partial y} \\ &\qquad + \frac{h_x^2\, h_y}{2} \frac{\partial^3 f}{\partial x^2\, \partial y} + \frac{h_x\, h_y^2}{2} \frac{\partial^3 f}{\partial x\, \partial y^2} + \ldots \,. \end{aligned}$$

Note that the contribution schema defined by equation (8) for the case of a single stock in a foreign currency is just an instance of the more general schema (48). Indeed in both cases the performance is completely explained by three contributions and no residual is present.

The projection operator It is useful to write equation (49) as an operator applied to the pricing function

f. To this end we define the *projection operator* with respect to the risk-driver x as

$$\Delta_x = \sum_{n=1}^{\infty} \frac{h_x^n}{n!} \frac{\partial^n}{\partial x^n} = h_x \frac{\partial}{\partial x} + \frac{h_x^2}{2!} \frac{\partial^2}{\partial x^2} + \dots \qquad (50)$$

and similarly for the y risk driver,

$$\Delta_y = \sum_{n=1}^{\infty} \frac{h_y^n}{n!} \frac{\partial^n}{\partial y^n} = h_y \frac{\partial}{\partial y} + \frac{h_x^2}{2!} \frac{\partial^2}{\partial x^2} + \dots, \qquad (51)$$

so that we can write

$$c^x = \Delta_x f \qquad \text{and} \qquad c^y = \Delta_y f\,. \qquad (52)$$

Remarkably using the operators Δ_x and Δ_y it is also possible to write the cross performance contribution as

$$c^{x*y} = [\Delta_x \circ \Delta_y]\, f\,. \qquad (53)$$

In this equation the symbol $\circ$ denotes the operator composition so that first we apply the right-most operator to the pricing function and then we apply the left-most operator to the result, i.e.,

$$[O_2 \circ O_1]\; f = O_2\, [O_1\, f]\;, \qquad (54)$$

where O_1 and O_2 are two generic operators.

Historical note We observe that the projection operators (see, for example, reference [9]), when applied to smooth pricing functions, can be written also as

$$\Delta_x = \sum_{n=1}^{\infty} \frac{h_x^n}{n!} \frac{\partial^n}{\partial x^n} = \exp\left\{ h_x \frac{\partial}{\partial x} \right\} - \mathbf{1}\,, \qquad (55)$$

and similar definition for the Δ_y operator. As an interesting historical note, the notation of the last term on the far right-hand side of equation (55) was used as early as the middle of the XIX century (see, for example, the book of Boole in reference [2]), and has the advantage of explicitly showing the non-linear dependence of the return operator from the displacement and the derivative operator. We do not use this notation, however, because it is significantly more cumbersome then the Δ_x notation and implicitly assumes the smoothness of the pricing function.

Properties of projection operators Using the one-dimensional complete Taylor expansion we obtain

$$\begin{aligned} \Delta_x f(x,y) &= \sum_{n=1}^{\infty} \frac{h_x^n}{n!} \frac{\partial^n}{\partial x^n} f(x,y) \\ &= \sum_{n=0}^{\infty} \frac{h_x^n}{n!} \frac{\partial^n}{\partial x^n} f(x,y) - f(x,y) \\ &= f(x+h_x) - f(x,y)\,, \end{aligned}$$

so that the operator Δ_x provides the return of pricing-function *projection* along the x axis (hence the name projection operator). In operator form the above equation can be written as

$$\Delta_x = \left[\sum_{n=0}^{\infty} \frac{h_x^n}{n!} \frac{\partial^n}{\partial x^n} - \mathbf{1} \right] = \mathbf{X} - \mathbf{1}\,,$$

where $\mathbf{X}$ is the x-axis translation operator defined as

$$[\mathbf{X} \circ f]\,(x,y) = f(x+h_x, y)\,.$$

For the y risk driver we can similarly define $\mathbf{Y}$ as the y-axis translation operator, i.e.

$$[\mathbf{Y} \circ f]\,(x,y) = f(x, y+h_y)\,.$$

Hence we can write the projection operators in terms of the translation operators as

$$\Delta_x = \mathbf{X} - \mathbf{1} \quad \text{and} \quad \Delta_y = \mathbf{Y} - \mathbf{1}\,, \qquad (56)$$

so that we equivalently have

$$\mathbf{X} = \mathbf{1} + \Delta_x \quad \text{and} \quad \mathbf{Y} = \mathbf{1} + \Delta_y\,. \qquad (57)$$

Note that by using definitions (50) and (51) for the projection operators we have to ensure the pricing function to be infinitely smooth. However if we were to use equations (56) as the definition of Δ_x and Δ_x, we could relax the smoothness assumption and allow for a more generic class of pricing functions. Therefore from now on we assume definitions (56), which are indeed equivalent to equations (57).

By using equations (56) as the definitions of the projection operators, we can take advantage of the translation-operator properties (see, again, reference

[9]) and those of the identity operator, to derive the traits of the projection operators. The first property we can deduce is the linearity of Δ_x and Δ_y, so that for any two pricing functions f and g and two constants A and B we have

$$\Delta_x\left(A\cdot f+B\cdot g\right)=A\cdot\Delta_x f+B\cdot\Delta_x g\,. \tag{58}$$

Also notice that for a pricing function linear in its variable,

$$f(x)=x \quad\Rightarrow\quad \Delta_x f(x)=h_x\,,$$

where h_x is the risk-driver displacement:

$$h_x=x'-x\,.$$

Since $\mathbf{X}$ and $\mathbf{Y}$ commute and they both commute with the identity $\mathbf{1}$ operator we also have

$$\Delta_x\circ\Delta_y=\Delta_y\circ\Delta_x\,,$$

so that the projection operators with respect to two different variables can be computed regardless of the risk-driver order. Finally for any pricing function $f(y)$ depending only on the second risk driver and $g(x)$ depending only from the first risk driver we have

$$\Delta_x\, f=0 \qquad\text{and}\qquad \Delta_y\, g=0\,. \tag{59}$$

As it happens, because of these equations any parameter that does not vary during the observation period (for example the strike in an equity option, or the coupon of a fixed-rate bond) yields a null contribution to the performance.

Contributions in terms of projection operators Using the translation operators $\mathbf{X}$ and $\mathbf{Y}$ we can write the exact contribution to performance (52) and (53) as

$$c^x=\left[\mathbf{X}-\mathbf{1}\right]f\,,\quad c^y=\left[\mathbf{Y}-\mathbf{1}\right]f\,, \tag{60}$$

and

$$c^{x*y}=\left[\mathbf{X}-\mathbf{1}\right]\circ\left[\mathbf{Y}-\mathbf{1}\right]f\,. \tag{61}$$

Therefore the cash performance can be written as

$$\begin{aligned}[\mathbf{X}\circ\mathbf{Y}-\mathbf{1}]\,f&=\left[\mathbf{X}-\mathbf{1}\right]f\\&\quad+\left[\mathbf{Y}-\mathbf{1}\right]f+\left[\mathbf{X}-\mathbf{1}\right]\circ\left[\mathbf{Y}-\mathbf{1}\right]f\,,\end{aligned} \tag{62}$$

i.e. the algebraic identity

$$\begin{aligned}f(x+h_x,y+h_y)-f(x,y)&=\\&\left[f(x+h_x,y)-f(x,y)\right]\\&\quad+\left[f(x,y+h_y)-f(x,y)\right]\\&+\left[f(x+h_x,y+h_y)-f(x+h_x,y)\right.\\&\qquad\left.-f(x,y+h_y)+f(x,y)\right]\,.\end{aligned} \tag{63}$$

Since equation (62) holds for a generic pricing function f we can write it in operator form as

$$[\mathbf{X}\circ\mathbf{Y}-\mathbf{1}]=\left[\mathbf{X}-\mathbf{1}\right]+\left[\mathbf{Y}-\mathbf{1}\right]+\left[\mathbf{X}-\mathbf{1}\right]\circ\left[\mathbf{Y}-\mathbf{1}\right]\,, \tag{64}$$

to be compared with the algebraic equation (9). In turn this equation can be written in terms of the projection operators as

$$[\mathbf{X}\circ\mathbf{Y}-\mathbf{1}]=\Delta_x+\Delta_y+\Delta_y\circ\Delta_x\,, \tag{65}$$

which in words means *the full performance operator is given by the sum of the projection operator along each axis plus the compound projection operator*. Moreover the risk-driver performance components given by equations (60) can be thought to be a generalization of the simple returns defined in equations (4) and (5). Similarly, the cross performance contribution given by equation (61) can be likend to the cross return defined in equation (7) due to the compounding of two risk drivers. In the case of a generic pricing function the term c^{x*y} may not be viewed as a traditional compounding, usually expressed as a mathematical multiplication, however it can still interpreted as the extra contribution resulting from the *composition* of the two operators Δ_x and Δ_y.

In an alternative approach to the definition of the projection contribution schema, we could have taken the shortcut of assuming the identity (63) as the axiomatic definition of the various performance components. While that approach seems reasonable from a purely mathematical point of view, it does not reveal what is the financial meaning of the cross terms: i.e. that they provide a contribution to performance that *cannot* be associated to a single risk driver. In the simple case of multiplicative pricing functions these contributions are purely compounding components due to the arithmetic, as opposed to geometric, nature of attribution. However, in the generic pricing-function framework they can be explained by the need to compound the return operators in order to obtain the exact performance.

3.4 Pricing functions depending on several risk drivers

We now consider the computation of the performance contributions for a portfolio depending on several risk drivers. We use the results obtained for a multiplicative pricing function of several variables to build a performance-contribution schema for a portfolio described by a generic pricing function depending on n risk drivers. Using this technique we are going to show that the risk-driver contribution, defined in equation (30), can be split in $2^n - 1$ terms as

$$c^d = c^{\text{drivers}} + c^{\text{pairs}} + c^{\text{triplets}} + \ldots + c^{n\text{-plet}} , \quad (66)$$

to be compared with equation (15). In this equation the term c^{drivers} can be further split as the sum of n contributions provided by each single risk driver, the term c^{pairs} is the sum of the $\binom{n}{2}$ contributions coming from all distinct risk-driver pairs, c^{triplets} is the sum of the $\binom{n}{3}$ contributions coming from all distinct risk-driver triplets, and so on and so forth. The reader not concerned with the mathematical proof of this equation can skip the rest of this subsection.

To prove equation (66) consider a pricing function $f(x_1, \ldots, x_n)$ depending on n risk drivers $x_1, \ldots, x_n$. Given any risk driver x_i, with $i \in \{1, \ldots, n\}$, and its displacement h_i, we define the i-th translation operator $\mathbf{X}_i$ as

$$[\mathbf{X}_i \circ f]\,(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, x_i + h_i, \ldots, x_n) ,$$

i.e. the map from a pricing function to its h_i-translation on the x_i axis. Notice that composing the first two translation operators, $\mathbf{X}_1$ and $\mathbf{X}_2$, we obtain

$$[\mathbf{X}_1 \circ \mathbf{X}_2 \circ f]\,(x_1, x_2, x_3, \ldots, x_n) = f(x_1 + h_1, x_2 + h_2, x_3, \ldots, x_n) ,$$

and that composing all n translation operators $\mathbf{X}_i$'s, recalling the definitions (45), we have

$$[\mathbf{X}_1 \circ \mathbf{X}_2 \circ \ldots \circ \mathbf{X}_n \circ f]\,(x_1, x_2, x_3, \ldots, x_n) = f(x_1', x_2', x_3', \ldots, x_n') ,$$

i.e. the pricing function evaluated with the risk drivers observed at the end of the period. We then define the i-th projection operator as

$$\Delta_i = \mathbf{X}_i - \mathbf{1} \qquad \text{for } i = 1, \ldots, n ,$$

where $\mathbf{1}$ is the identity operator defined in equation (47). Note that all the operators Δ_i's are defined on a wide range of functions, not just the infinitely smooth pricing functions.

The risk-driver contribution to the portfolio cash performance can then be written in terms of the composition of the n translation operators to which we need to subtract the identity:

$$\begin{aligned} c^d &= f(x_1', \ldots, x_n') - f(x_1, \ldots, x_n) \\ &= [(\mathbf{X}_1 \circ \ldots \circ \mathbf{X}_n) - \mathbf{1}]\, f \\ &= \left[m_n(\mathbf{X}_1, \ldots, \mathbf{X}_n)\right] f . \end{aligned}$$

In the above expression, $m_n(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ denotes the multiplicative pricing function, defined by equation (16), computed however as a function of the n translation operators $\mathbf{X}_i$'s replacing the usual real-number multiplication $\cdot$ with the operator composition $\circ$ defined by equation (54) and the number 1 with the identity operator $\mathbf{1}$.

Because of the linearity of the projection operators, we can split m_n as in equation (20) and write C^d as a sum of contributions, i.e.

$$c^d = \sum_{i=1}^{n} C^{\mathrm{X}_i} f + \sum_{k=2}^{n} \sum_{\{j_1, \ldots, j_k\} \in \mathcal{C}_k^n} C^{\mathrm{X}_{j_1} * \mathrm{X}_{j_2} * \ldots * \mathrm{X}_{j_k}} f , \quad (67)$$

where the risk-driver terms are given by

$$c^{x_i} = C^{\mathrm{X}_i} f = [\mathbf{X}_i - \mathbf{1}]\, f = \Delta_i f ,$$

and the cross contributions are computed using equations (18):

$$\begin{aligned} c^{x_{j_1} * \ldots * x_{j_k}} &= C^{\mathrm{X}_{j_1} * \ldots * \mathrm{X}_{j_k}}\, f \\ &= \left\{ \left[\mathbf{X}_{j_1} - \mathbf{1}\right] \circ \ldots \circ \left[\mathbf{X}_{j_k} - \mathbf{1}\right] \right\} f \\ &= \left[\Delta_{j_1} \circ \ldots \circ \Delta_{j_k}\right] f . \end{aligned} \quad (68)$$

We categorize the performance contributions in equation (67) according to the degree of the polynomials in the projection operators. Accordingly we call first-degree terms the contributions coming from a single risk driver. Similarly we say that the contributions coming from two risk drivers are of second degree, those from three risk drivers of third degree and so on and so forth.

Note that equation (67) can also be written as expression (66), thus concluding the proof of that formula. Similarly to the two-dimensional case of equation (65), the definition of the cross components (68) suggests that these terms are due to non-linear compounding effects of many risk drivers at the same time. Again, these effects cannot be explained by the variations of any single risk driver by itself.

Since it was obtained using composition of projection operators we say that equation (66) defines the projection contribution schema: i.e. the main result of this paper.

3.5 Simplified projection schemata

The projection contribution schema defined by equation (66) has the advantage of being very detailed in describing the different contributions to performance. One drawback of such a schema is that it provides too much information and that is very expensive to compute. In order to deal with a smaller number of contributions we can, for example, gather together all the terms that are obtained as the compounding of three or more risk drivers. In this way we define the triplet-or-higher term as

$$c^{\text{triplet-or-higher}} = c^{\text{triplets}} + \ldots + c^{n\text{-plet}} ,$$

so that we obtain the *second-degree projection schema* defined as

$$c^d = c^{\text{drivers}} + c^{\text{pairs}} + c^{\text{triplet-or-higher}} . \qquad (69)$$

In practical computations the term $c^{\text{triplet-or-higher}}$ is cheaply computed by inverting equation (69) as

$$c^{\text{triplet-or-higher}} = c^d - c^{\text{drivers}} - c^{\text{pairs}} .$$

When even the second-degree projection schema provides too many details we can gather together all the terms obtained by the compounding of two or more risk drivers into the term

$$c^{\text{pairs-or-higher}} = c^{\text{pairs}} + c^{\text{triplets}} + \ldots + c^{n\text{-plet}} ,$$

and define the *first-degree projection schema* as

$$c^d = \sum_{i=1}^{n} c^{x_i} + c^{\text{pairs-or-higher}} , \qquad (70)$$

where

$$c^{x_i} = f(\ldots, x_i + h_i, \ldots) - f(\ldots, x_i, \ldots) ,$$

is i-th the risk-driver contribution. Again, in practice, one usually computes $c^{\text{pairs-or-higher}}$ as

$$c^{\text{pairs-or-higher}} = c^d - \left(\sum_{i=1}^{n} c^{x_i} \right) .$$

Depending on the case at hand the portfolio manager may decide to use the full projection schema, the second degree one, or settle for the less detailed results obtained using the first-degree projection schema.

4 Practical examples of performance attribution

In this section we provide some practical cases for the computation of the projection-contribution schema outlined in the last section.

4.1 Explicit computations for three risk drivers

The projection-contribution schema defined by equation (66) may seem, at first, a little too theoretical and disconnected from the actual day-to-day performance computations. In order to show how to use that formula we derive explicitly the contributions to c^{drivers}, c^{pairs}, and c^{triplets}, for a pricing function $f(x, y, z)$ depending on three risk drivers x, y, and z. The first term c^{drivers} is the sum of the single risk-driver contributions, i.e.

$$c^{\text{drivers}} = c^x + c^y + c^z ,$$

worked out in equations (60) for c^c and c^y, and similarly for the z contribution:

$$\begin{aligned} c^x &= f(x+h_x, y, z) - f(x, y, z)\,, && (71)\\ c^y &= f(x, y+h_y, z) - f(x, y, z)\,, && (72)\\ c^z &= f(x, y, z+h_z) - f(x, y, z)\,. && (73) \end{aligned}$$

The pair contribution c^{pairs} can be written as the sum of the pair cross contributions,

$$c^{\text{pairs}} = c^{x*y} + c^{x*z} + c^{y*z}\,,$$

where the cross contribution c^{x*y} for the two risk drivers x and y can be computed from equation (61) as

$$\begin{aligned} c^{x*y} &= C^{X*Y} f = \left[\mathbf{Y}-\mathbf{1}\right] \circ \left[\mathbf{X}-\mathbf{1}\right] f\\ &= \left[\mathbf{X}\circ\mathbf{Y} - \mathbf{X} - \mathbf{Y} + \mathbf{1}\right] f\\ &= f(x+h_x, y+h_y, z) - f(x+h_x, y, z) \qquad (74)\\ &\qquad - f(x, y+h_y, z) + f(x, y, z)\,, \end{aligned}$$

and similarly for the two other pairwise cross contributions

$$\begin{aligned} c^{x*z} = f(x+h_x, y, z+h_z) - f(x+h_x, y, z) \quad (75)\\ - f(x, y, z+h_z) + f(x, y, z)\,, \end{aligned}$$

and

$$\begin{aligned} c^{y*z} = f(x, y+h_y, z+h_z) - f(x, y+h_y, z) \quad (76)\\ - f(x, y, z+h_z) + f(x, y, z)\,. \end{aligned}$$

Finally, since n=3, the triplet contribution has only one component,

$$c^{\text{triplets}} = c^{x*y*z}\,,$$

arising from the co-movements of the three risk drivers x, y, and z together. We can calculate this term explicitly with the help of equation (14):

$$\begin{aligned} c^{x*y*z} &= [\mathbf{Z}-\mathbf{1}] \circ [\mathbf{Y}-\mathbf{1}] \circ [\mathbf{X}-\mathbf{1}]\ f\\ &= [\mathbf{Z}\circ\mathbf{X}\circ\mathbf{Y} - \mathbf{X}\circ\mathbf{Y} - \mathbf{Z}\circ\mathbf{X}\\ &\qquad - \mathbf{Z}\circ\mathbf{Y} + \mathbf{X} + \mathbf{Y} + \mathbf{Z} - \mathbf{1}]\, f\,, \end{aligned}$$

so that,

$$\begin{aligned} c^{x*y*z} = \qquad\qquad (77)\\ f(x+h_x, y+h_y, z+h_z) - f(x+h_x, y+h_y, z)\\ - f(x+h_x, y, z+h_z) - f(x, y+h_y, z+h_z)\\ + f(x+h_x, y, z) + f(x, y+h_y, z)\\ + f(x, y, z+h_z) - f(x, y, z)\,. \end{aligned}$$

4.2 Projection schema for an equity option

In this subsection we consider the example of an equity vanilla option. Even though this case is very specific the conclusions are typical of any equity option: the performance depends on changes both in the underlying spot price and its volatility together with the additional non-trivial contribution coming from the cross interaction of price and volatility. As may be intuitive the changes in the interest rates provide smaller effect than those just described.

Consider the gain, or the loss, of an equity vanilla call option, on a non-paying-dividend stock, in the local currency, expiring at a time T with the following parameters:

$T-t$	Δt	strike
0.5	7/365	100

together with the following quotes for the risk drivers and the option premium:

	s	r	σ	f
t	100	0.98%	25%	7.273
t'	87	1.14%	33%	3.668

We evaluate the option premium using the standard Black-Scholes-Merton formula so that the pricing function is given by equation (24). In order to explain a cash performance[1] Δf in terms of the variations of the option risk drivers s, r, and σ we can proceed with the method outlined in the previous section using the explicit formulas of subsection 4.1.

According to the standard perturbational-attribution model, described by equation (2), we would

[1] In this case the negative signs actually indicates a loss.

have

$$\begin{aligned}\Delta f = &\quad c^t \quad + \quad c^s \quad + \quad c^r \quad + \quad c^\sigma \quad + \quad \varepsilon \\ = &\ -0.143 \ - 5.671 + 0.038 + 2.242 - 0.535 \\ = &\ -3.605\,,\end{aligned} \tag{78}$$

where the delta and the gamma contributions have been grouped together to create the total equity contribution c^s:

$$c^s = c^\Delta + c^\Gamma = -7.101 + 1.894 = -5.671\,.$$

As mentioned in the introduction the main problem with this attribution is that the residual term, in absolute value, is significantly bigger than both the time contribution c^t and the interest-rate contribution c^r. Therefore it is fair to conclude that we are missing some important contributions to the performance.

In order to find the missing contributions we compute the exact performance components as outlined in section 3 using the results of subsection 4.1 for a pricing function of three variables, setting $x = s$, $y = r$, and $z = \sigma$. For convenience we split the performance as

$$\Delta f = c^t + c^{\text{drivers}} + c^{\text{pairs}} + c^{\text{triplets}}\,. \tag{79}$$

Since options do not pay coupons we have c^t_{acc}=0, so that the time contribution,

$$c^t = c^t_{\text{conv}} = -0.145\,,$$

is entirely given by the convergence term (and it is negative as expected). Then we consider the contributions coming separately from the three risk drivers that can be computed using equations (71)-(73):

$$\begin{aligned}c^{\text{drivers}} = &\quad c^s \quad + \quad c^r \quad + \quad c^\sigma \\ = &\ -5.111 \ + \ 0.037 \ + \ 2.196 \\ = &\ -2.878\end{aligned}$$

We note that, similarly to the perturbational approach, we have a dominant negative contribution from the equity price, a positive contribution coming from the increased volatility, and a small contribution from interest-rate variations.

The cross pair contributions to performance can be computed using equations (74)-(76):

$$\begin{aligned}c^{\text{pairs}} = &\quad c^{s*r} \quad + \quad c^{s*\sigma} \quad + \quad c^{r*\sigma} \\ = &\ -0.022 \ - \ 0.564 \ - \ 0.001 \\ = &\ -0.578\,,\end{aligned}$$

where we notice two small contributions from the pairs of risk drivers that include the interest rate and a significant contribution coming from the interaction of equity/volatility variations.

Finally we have a tiny contribution coming from the triplet interaction of the interest rate, the equity price, and the volatility:

$$c^{\text{triplets}} = c^{s*r*\sigma} = 0.005\,.$$

With this contribution all terms in equation (79) are accounted for and no residual is necessary to explain the performance. At this point we notice that the traditional perturbative approach to the attribution, see equation (78), has a residual ε=-0.535 very similar in size to the price/volatility cross contribution $c^{s*\sigma}$=-0.564, which must be the missing piece we were looking for.

4.3 Portfolio of fixed-rate bonds with credit risk

As an example of a model pricing function depending on four risk drivers we consider a portfolio composed of an arbitrary number of fixed-rate coupon bonds with credit risk. For ease of discussion we collect together all cash flows and split them according to their maturity: short-dated cash flows, denoted as a_j's, are discounted at a rate y_1, long-dated ones, denoted as b_k's, are discounted using a rate y_2. Similarly we use a credit spread s_1 for the short-dated coupons and a credit spread s_2 for the long-dated ones. In total there are n=4 risk drivers: y_1, y_2, s_1, and s_2. We also assume that the only current coupon is a_1, accruing from T_0 to T_1. The continuously-compounded pricing function[2], see equation (31), can be written in this case is

$$f(y_1, y_2, s_1, s_2; t) = \sum_j a_j\, e^{-(y_1+s_1)\,(T_j-t)} + \sum_k b_k\, e^{-(y_2+s_2)\,(T_k-t)}\,, \tag{80}$$

[2] We could also use the annually-compounded discount, however, obtaining more cumbersome results.

where all bond redemptions have been included in the cash flows a_j's and b_k's.

The computation of the accrual term is straightforward, since f^a is defined exactly as in equation (40), so that the term c^t_{acc} can be computed using equation (41). The convergence term can be computed exactly as the remaining part of the time contribution:

$$c^t_{\mathrm{conv}} = \left[e^{(y_1+s_1)\,\Delta t} - 1\right] \cdot \sum_j a_j\, e^{-(y_1+s_1)\,(T_j-t)} + \left[e^{(y_2+s_2)\,\Delta t} - 1\right] \cdot \sum_k b_k\, e^{-(y_2+s_2)\,(T_k-t)} - a_1 \frac{t-T_0}{T_1-T_0} \,.$$

As remarked earlier since n=4 there is in principle a total of 2^4-1=15 risk-driver contributions, however, we are going to show that many of these components are exactly zero. The computation of the performance contributions can be simplified by noticing that for a pricing function of the form

$$f(x) = e^{g(x)} \,,$$

we have

$$\begin{aligned} \Delta_x f &= e^{g(x')} - e^{g(x)} \\ &= e^{g(x)} \left[e^{g(x')-g(x)} - 1\right] \\ &= e^{g(x)} \left[e^{[\Delta_x \circ g](x)} - 1\right] . \end{aligned}$$

In a more symbolic notation we can write the above rule as

$$\Delta_x[e^g] = g \cdot \left(e^{\Delta_x g} - 1\right) .$$

Using this rule and the operator linearity, see equation (58), the contributions of the single risk drivers to the performance can computed as

$$\begin{aligned} c^{y_1} &= \Delta_{y_1} f \\ &= \sum_j a_j\, e^{-(y_1+s_1)\,(T_j-t')} \left[e^{-(T_j-t')\Delta y_1} - 1\right] , \\ c^{y_2} &= \Delta_{y_2} f \\ &= \sum_k b_k\, e^{-(y_2+s_2)\,(T_k-t')} \left[e^{-(T_k-t')\Delta y_2} - 1\right] , \\ c^{s_1} &= \Delta_{s_1} f \\ &= \sum_j a_j\, e^{-(y_1+s_1)\,(T_j-t')} \left[e^{-(T_j-t')\Delta s_1} - 1\right] , \\ c^{s_2} &= \Delta_{s_2} f \\ &= \sum_k b_k\, e^{-(y_2+s_2)\,(T_k-t')} \left[e^{-(T_k-t')\Delta s_2} - 1\right] , \end{aligned}$$

where

$$\Delta y_1 = y_1' - y_1 \,, \qquad \Delta y_2 = y_2' - y_2 \,,$$

$$\Delta s_1 = s_1' - s_1 \,, \qquad \text{and} \qquad \Delta s_2 = s_2' - s_2 \,.$$

There are, in principle, $\binom{4}{2} = 6$ pair components however only two of them are not null:

$$\begin{aligned} c^{s_1 * y_1} &= [\Delta_{s_1} \circ \Delta_{y_1}]\, f \\ &= \sum_j a_j\, e^{-(y_1+s_1)\,(T_j-t')} \,. \\ &\quad \cdot \left[e^{-(T_j-t')\Delta y_1} - 1\right] \cdot \left[e^{-(T_j-t')\Delta s_1} - 1\right] \end{aligned}$$

and

$$\begin{aligned} c^{s_2 * y_2} &= [\Delta_{s_2} \circ \Delta_{y_1}]\, f \\ &= \sum_k b_k\, e^{-(y_2+s_2)\,(T_k-t')} \,. \\ &\quad \cdot \left[e^{-(T_k-t')\Delta y_2} - 1\right] \cdot \left[e^{-(T_k-t')\Delta s_2} - 1\right] . \end{aligned}$$

In order to compute the other four components we note that using property (59) it can be shown that

$$c^{y_1 * y_2} = 0 \,, \qquad c^{s_1 * y_2} = 0 \,,$$

$$c^{y_1 * s_2} = 0 \,, \qquad \text{and} \qquad c^{s_1 * s_2} = 0 \,.$$

Finally, for the same reason, all four triplet contributions and the quadruplet contribution are identically zero.

Collecting only the non-zero contributions, we can write the performance portfolio for a portfolio of fixed-rate coupon bonds as

$$\begin{aligned}\Delta f = & \; c^t_{\text{acc}} + c^t_{\text{conv}} \\ & + c^{y_1} + c^{y_2} + c^{s_1} + c^{s_2} \\ & + c^{s_1 * y_1} + c^{s_2 * y_2} .\end{aligned} \tag{81}$$

Observe that this result is exact and that there is no need for any of the terms Δt, Δy_1, Δy_2, Δs_1,or Δs_2, to be small.

Grouping risk drivers We note that the risk drivers for the bond pricing function (80) can be naturally split into two groups: the *interest-rate group* and the *credit-spread group*. The interest-rate group, denoted as $\mathbf{y}$, is composed by the two interest rates y_1 and y_2, while the credit-spread group, denoted as $\mathbf{s}$, has the two credit spreads z_1 and z_2 as constituents. We can then define the group risk-driver contributions as

$$c^{\mathbf{y}} = c^{y_1} + c^{y_2} , \qquad \text{and} \qquad c^{\mathbf{s}} = c^{s_1} + c^{s_2} , \tag{82}$$

and the group cross contribution as

$$c^{\mathbf{s}*\mathbf{y}} = c^{s_1 * y_1} + c^{s_2 * y_2} . \tag{83}$$

With this definitions and equation (36), the projection schema (81) can be written in the simplified form

$$\Delta f = c^t + c^{\mathbf{y}} + c^{\mathbf{s}} + c^{\mathbf{s}*\mathbf{y}} . \tag{84}$$

More in general, when a portfolio depends on a large numer of risk drivers, and some of these risk drivers have some common property, it is convenient to consider groups of risk drivers and aggregate the different contributions accordingly.

4.4 PIIGS crisis bond portfolio

In this subsection we provide a practical example useful to apply the results of the previous subsection. We look at the PIIGS crisis of 2011/2012 for a small portfolio of fixed-rate Italian government bonds composed by two short-dated securities and a longer maturity one. In the typical attribution approach we need to compute the portfolio components and compare them to the reference benchmark: the German treasury yield curve. Hence we are not just interested in the performance of the Italian-treasury bonds per se, however, in their extra performance due to the spread between the Italian and the German yield curves. Furthermore we are interested in how much of this performance can be attributed to the short maturities and how much to the long ones.

Even thought an actual portfolio typically contains dozens of bonds, we build a simple representative portfolio with the Italian treasury notes defined in table[3] 1. All bonds pay coupons semiannually according to the *Act/Act* day-count convention. We build a portfolio containing one security of nominal 100 for each of these bonds. We also assume that all cash flows up to two years in the future are discounted, continuously compounded, with a yield $y_1 + s_1$ and that all cash flows further than two years are discounted at $y_2 + s_2$. In computing the discount factor, the yields y_1 and y_2 are observed from the benchmark, i.e. the German Treasury yield curve, while s_1 and s_2 are the spread between the benchmark and the Italian yield curve respectively at two and ten years.

The performance is computed for the period that spans the first five months of the bond issue, i.e. from the end of November 2011 to the end of April 2012. We use the market data in table 2, where all yields and spreads are quoted with an *Act/365* day-count convention and are continuously compounded. From this table we can compute the yield displacements:

$$\Delta y_1 = -0.349\% , \qquad \Delta y_2 = -0.704\% ,$$

and the spread displacements:

$$\Delta s_1 = -3.908\% , \quad \text{and} \quad \Delta s_2 = -1.396\% .$$

[3] To simplify the computations these are not actually traded bonds, however they are very plausible.

description	issue date	maturity	coupon	f	f'
BOT 1Y	2011-11-29	2012-11-29	0%	92.829	98.171
BTP 2Y	2011-11-29	2013-11-29	2%	99.929	110.700
BTP 10Y	2011-11-29	2021-11-29	6.5%	91.305	109.569

Table 1: List of Italian virtual bonds considered in the example of subsection 4.4.

curve node	date	$y_i + s_i$ (%)	y_i (%)	s_i (bips)
i=1	29-Nov-11	7.421	0.438	698.3
i=2	29-Nov-11	7.618	2.362	525.6
i=1	30-Apr-12	3.164	0.089	307.5
i=2	30-Apr-12	5.518	1.658	386.0

Table 2: Market data used to compute the performance attribution of subsection 4.4.

We organize the results as follows: first we provide the aggregate result, see equation (84), i.e.

$$\begin{aligned}\Delta f &= r^t + r^{\mathbf{y}} + r^{\mathbf{s}} + r^{\mathbf{y}*\mathbf{s}}\\ &= 3.19\% + 2.05\% + 6.62\% + 0.45\%\\ &= 12.10\%\,,\end{aligned} \tag{85}$$

then we look into the details of each these terms. Here and in the rest of the section, we use equation (27) to transform the cash variations in percentage returns.

Since both the benchmark interest rates and the credit spreads decreased during the evaluation period, it is not surprising that all performance components in equation (85) are positive. A somewhat unexpected result is the contribution of 45 basis points, i.e. almost half of a percentage point, to the performance due to the combined effects of interest rates and spreads. This contribution can be thought to be due to the compounding effects of interests and credit.

As in most fixed-income portfolios, the carry term provides a significant chunk of performance and can be split, see equation (36), into a coupon-accrual term and a convergence term:

$$r^t = r^t_{\text{acc}} + r^t_{\text{conv}} = 1.26\% + 1.93\%\,.$$

The yield contribution to performance, as seen in the first of equations (82), can be split into a short-term component r^{y_1}, for cash flows with a maturity less than two years, and a long-term component r^{y_2}, for cash flows paid after two years:

$$r^{\mathbf{y}} = r^{y_1} + r^{y_2} = 0.26\% + 1.79\%\,.$$

Observe how the short-term component r^{y_1} is about half of the cross yield/spread component $r^{\mathbf{s}*\mathbf{y}}$, a fact that emphasizes even more the importance of the co-movements of spreads and yields.

The spread component as well, as seen in the second equation of (82), can be split into short-term and long-term contributions,

$$r^{\mathbf{s}} = r^{s_1} + r^{s_2} = 2.96\% + 3.67\%\,,$$

where, as expected, both components strongly contribute to the global performance.

At last the group cross contribution can also be split into short and long components, refer to equation (83) for its computation, so that

$$r^{\mathbf{s}*\mathbf{y}} = r^{s_1*y_1} + r^{s_2*y_2} = 0.01\% + 0.23\%\,.$$

Notice that the short-maturity *yield-cross-spread* component $r^{s_1*y_1}$, as tiny as it is, is however a fundamental part of the performance when all basis points need to be accounted for so that no residual leftover has to be explained.

5 Summary and conclusions

In this paper we define a projection-contribution schema useful to compute the risk-driver components for the performance of non-linear portfolios. We consider a single period, not necessarily short, in which all dividends and coupons are re-invested in the portfolio itself so that no cash flows are coming in or out of the portfolio. More importantly we require that there exists a computable pricing function that approximates the portfolio value at the beginning and at the end of the period. Under these conditions we separate the performance into two main components: the first one depending solely on the passage of time and the second one depending on the variations of the underlying risk drivers at the end of the period.

Furthermore we find that it is possible to exactly split the risk-driver component into 2^n-1 contributions: n risk-driver contributions, $\binom{n}{2}$ pair contributions, $\binom{n}{3}$ triplets contributions, and so on and so forth until the last contribution coming from the co-movements of all risk drivers together. In the case of pricing functions multiplicative in their arguments the cross components can simply be explained by the geometric compounding of risk drivers. Similarly, in the generic case the cross terms can be interpreted as due to the compounding of the projection operators from different risk drivers.

Many of the cross contributions are typically small and can be neglected. It often turns out that some of the cross-driver contributions are exactly zero as seen, for example, in subsection 4.3. We also show how it may be convenient to aggregate risk drivers that share a common property (for examples yields belonging to the same interest-rate term structure) and sum the corresponding contributions into driver-group components. When a simplified version of the risk-driver dependence is sufficient, we can use the first-degree or the second-degree projection schema as defined in subsection 3.5.

We also examine some practical applications in two examples: an equity vanilla option and a portfolio of government bonds. In the vanilla-option case we find that the cross term between the spot price and its volatility is larger than any of the other contributions, simple or cross, coming from the interest-rate variations. Naturally the same method can be applied to any type of options such as barrier options, digital options, and so on. Since options usually do not pay cash flows it is possible to employ the same method to split the performance for any time period, even from the first day the option is traded to its maturity date.

The government-bond portfolio allows us to attribute the performance to interest rates and credit spreads for long and short maturities. In this case we find that many cross terms are identically zero and that, again, the full performance could be explained when all the components were considered.

Once again we stress that, in the framework described in this paper, all contributions are computed independently from any Taylor approximation and are exact. In some cases, notably when an analytical pricing function is available such as in the case of the fixed-rate coupon bond, it is even possible to provide analytical formulas for the performance components (see subsection 4.3). Another important advantage of the first-principle approach with respect to the perturbative method is that it is applicable even to portfolios that are delta or gamma hedge. Indeed for those portfolios the traditional perturbative attribution fails since first and second-order sensitivities are usually very small.

Sometimes in asset management, see for example reference [8], one approximates the full portfolio pricing function, depending on a large number n of risk drivers $x_1, \ldots, x_n$, with another much simpler factor function $\Phi(\xi_1, \ldots, \xi_k)$ depending on k risk factors, with $k \ll n$. In this cases it is still possible use the projection schema (full or simplified) to the risk-factor pricing function Φ, obtaining a time contribution, k risk-factor contributions, $\binom{k}{2}$ contributions coming from the risk-factor pairs, and so on and so forth.

Finally, as seen in reference [7], the projection schemata defined in this paper can be used as the basis for the computation of risk components, providing the portfolio manager with projection risk contributions that are consistent with the corresponding performance decomposition.

A Perturbative contribution schema

In this appendix we introduce the *perturbative contribution schema* that can be used when the pricing function is known only in terms of its first and second derivatives.

A-1 Perturbational calendar contribution

In subsection 3.1 we define the calendar performance component as in equation (28). When the elapsed period Δt is small the time contribution to performance can be approximated using the first-order Taylor approximation:

$$c^t \simeq \Delta t \frac{\partial f}{\partial t},$$

where the elapsed time Δt is defined as

$$\Delta t = t' - t.$$

In the case of a portfolio composed only by plain/vanilla options, such as that described by the pricing function (24), we have

$$c^t \simeq -\Delta t \cdot \Theta, \tag{A-1}$$

where Θ is the usual option Greek *theta*, i.e. the sum of the option first-order derivative with respect to the time to maturity $T - t$. The minus sign in equation (A-1) arises because of the change of variable from the current time to the time-to-maturity: $t \to T - t$.

For the fixed-rate coupon bond with a pricing function described by equation (23), the carry contribution can be approximated by

$$c^t \simeq \Delta t \left[\frac{a_1 \ln(1+y_1)}{(1+y_1)^{(T_1-t)}} + \ldots + \frac{a_n \ln(1+y_n)}{(1+y_n)^{(T_n-t)}} \right]. \tag{A-2}$$

Notice that for a small Δt, i.e. in the limit $t' \to t$, we have

$$(1+y_j)^{\Delta t} - 1 \to \Delta t \ln(1+y_j) \qquad \text{for } j = 1, \ldots, n.$$

When the yield term structure is flat, i.e. we have $y = y_i$ for all $i = 1, \ldots, n$, we can write

$$c^t \simeq \Delta t \cdot f \cdot \ln(1+y),$$

a term that *does not* converges to equation (34). If we further assume the yield to maturity y to be small we can use the Taylor approximation $\ln(1+y) \simeq y$ to obtain

$$c^t \simeq y \cdot f \cdot \Delta t, \tag{A-3}$$

i.e. equation (34).

Note that in general for a pricing function $f(y;t)$ to satisfy equation (34) we must have

$$\frac{\partial f(y;t)}{\partial t} = y \cdot f(y;t),$$

so that, solving this ordinary differential equation in f, we have

$$f(t) = K(y) \cdot e^{y\,t}, \tag{A-4}$$

for any choice of the yield function $K(y)$. Hence, for fixed-rate bonds, a sufficient condition for equation (A-3) to hold perturbatively in Δt, is to express bonds prices using continuously-compounded yields, i.e. using equation (31).

A-2 Perturbative contributions for two risk drivers

Most fixed-income-attribution frameworks available in literature are based on the perturbative expansion of the pricing function. This is the approach we adopt in this appendix, with the notable difference with respect to other works, that the pricing function, consistently with equation (30), should be evaluated at the end of the period, however, with the risk drivers observed at the beginning of the period.

First-order contributions for two risk drivers Consider a pricing function $f(x, y)$ of only two risk drivers x and y. We can write the performance first-order Taylor expansion as

$$c^d = f(x+h_x, y+h_y) - f(x,y) \simeq h_x \frac{\partial f}{\partial x} + h_y \frac{\partial f}{\partial y}, \tag{A-5}$$

with the risk-driver displacements defined as $h_x = x' - x$ and $h_y = y' - y$, so that

$$c^d = c^x + c^y + \varepsilon, \tag{A-6}$$

where ε is the residual and we have

$$c^x = h_x \frac{\partial f}{\partial x}, \qquad c^y = h_y \frac{\partial f}{\partial y}. \tag{A-7}$$

In this way we approximately split the performance into two contributions each coming from a risk driver. There are two major drawbacks of equation (A-6), the first limitation is that, as we have already seen in subsection 2.1, we are missing the contribution, possibly small, coming from the compounding of both risk drivers. The second weak point is that the neglected convexity terms very often provide significant performance contributions. In order to retrieve the cross component c^{x*y} and the other convexity terms we need to extend the Taylor approximation to the second order.

Second-order contributions for two risk drivers For a generic pricing function of two variables $f(x, y)$, smooth in its arguments, we can write the second-order Taylor expansion as

$$\begin{aligned} c^d &= f(x + h_x, y + h_y) - f(x, y) \\ &\simeq h_x \frac{\partial f}{\partial x} + h_y \frac{\partial f}{\partial y} \\ &\quad + \frac{h_x^2}{2} \frac{\partial^2 f}{\partial x^2} + \frac{h_y^2}{2} \frac{\partial^2 f}{\partial y^2} + h_x\, h_y \frac{\partial^2 f}{\partial x\, \partial y}\,, \end{aligned}$$

from which the contribution schema can be written as

$$c^d = c^x + c^y + c^{x*y} + \varepsilon\,. \tag{A-8}$$

In the above expression ε is the new residual,

$$c^x = h_x \frac{\partial f}{\partial x} + \frac{h_x^2}{2} \frac{\partial^2 f}{\partial x^2}\,, \tag{A-9}$$

and similarly for c^y; finally the cross term c^{x*y} is defined as

$$c^{x*y} = h_x\, h_y \frac{\partial^2 f}{\partial x\, \partial y}\,. \tag{A-10}$$

Observe that while many earlier authors keep the first-order and the second-order components separate, in equation (A-9) we merge them together because they both come from the displacements of the same risk driver. It is our belief that the convexity term *should not* be a separate component of the performance schema.

Consider as a practical example the case of a single stock in a foreign currency describe in subsection 2.1. In this case the pricing function is given by equation (21), i.e. $f(x, s) = s \cdot x$, where s and x are, respectively, the two risk drivers representing the stock price and the currency-exchange rate. The performance contributions can be explicitly computed as

$$c^x = (x' - x)\, s\,, \qquad c^s = (s' - s)\, x\,,$$

and

$$c^{x*s} = (x' - x)\,(s' - s)\,.$$

As a matter of fact, writing equation (27) for these terms we have

$$\begin{aligned} r^x &= \frac{c^x}{s \cdot x} = \frac{x' - x}{x}\,, \\ r^s &= \frac{c^s}{s \cdot x} = \frac{s' - s}{s}\,, \\ r^{x*s} &= \frac{c^{x*s}}{s \cdot x} = \frac{(x' - x) \cdot (s' - s)}{s \cdot x}\,, \end{aligned}$$

which are exactly[4] equations (4), (5), and (7) of subsection 2.1.

A-3 Second-order contributions from several risk drivers

The generalization to contributions for more than two variables is straightforward. For example for a generic pricing function depending on n risk drivers $x_1, \ldots, x_n$, we have

$$\begin{aligned} c^d &= f(x_1 + h_1, \ldots, x_n + h_n) - f(x_1, \ldots, x_n) \\ &= \sum_{i=1}^{n} c^{x_i} + \sum_{\{i,j\} \in \mathcal{C}_2^n} c^{x_i * x_j} + \varepsilon^{3\text{rd}}\,, \end{aligned} \tag{A-11}$$

with ε being the residual, the risk-driver contributions given by

$$c^{x_i} = h_i \frac{\partial f}{\partial x_i} + \frac{h_i^2}{2} \frac{\partial^2 f}{\partial x_i^2} \qquad \text{for } i = 1, \ldots, n\,,$$

[4] In this case we obtain the exact results because the pricing function, being quadratic in its arguments, is exactly equal to its own Taylor expansion to the second order.

and the pair contributions computed as

$$c^{x_i * x_j} = h_i\, h_j \frac{\partial^2}{\partial x_i \partial x_j} f\,, \qquad \text{(A-12)}$$

for $\{i,j\} \in \mathcal{C}_2^n$. We recall that the notation $\{i,j\} \in \mathcal{C}_2^n$ means that we consider all distinct pairs of indexes $\{i,j\}$ for i and j running from 1 to n. Again, all derivatives should be computed at the end of the period t' using the values of the risk drivers observed at the beginning of the period t.

In the example of a yearly-compounded fixed-rate coupon bond, as described by the pricing function (23), we have

$$c^{y_i} = -\frac{T_i - t'}{1+y_i} \frac{a_i\,\Delta y_i}{(1+y_i)^{(T_i-t')}} + \frac{(T_i - t')^2}{2\,(1+y_i)^2} \frac{a_i\,(\Delta y_i)^2}{(1+y_i)^{(T_1-t')}}\,,$$

for $i = 1, \ldots, n$, where we used the familiar notation $h_i = \Delta y_i$ and

$$c^{y_i * y_j} = 0 \qquad \text{for } \{i,j\} \in \mathcal{C}_2^n\,.$$

Note that for a flat yield curve, i.e. when $y_i = y$ for all i, we have

$$c^d = -M'_D\,\Delta y + \frac{\mathcal{C}'_V}{2}(\Delta y)^2 + \varepsilon\,,$$

where M'_D and $\mathcal{C}'_V$ are, respectively, the modified duration and the convexity computed using the end of the period time t' as the current time.

Using the perturbative contribution schema described by equation (A-11) we can provide an useful estimate of the performance contributions for a large range of portfolios for which the first and second pricing-function derivatives are known. Unfortunately, there are only few examples of pricing functions where it is possible to compute the portfolio derivatives analytically and in general one turns to numerical methods. For example, the first derivative with respect to a risk driver x_i can be computed using finite differences (see, e.g., reference [2]), as

$$h \cdot \frac{\partial f}{\partial x_i} \simeq f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)\,. \qquad \text{(A-13)}$$

The main trouble of the perturbative attribution just described is that, for most portfolio pricing functions, there is always a non-zero residual term ε^{3rd} that *does not* have a clear financial interpretation. Indeed, using the results of section 3, it is easy to show that the residual ε^{3rd} observed in equation (A-11) is a mix of terms coming from single risk drivers and compounding of risk drivers. Because of the presence of contributions from single risk drivers, ε^{3rd} *cannot* be dismissed as an interaction term, as it is often suggested in literature.

The term ε^{3rd} should be compared with the corresponding residual $c^{\text{triplet-or-higher}}$ of the second-degree projection schema described in equation (69). Differently from ε^{3rd}, the term $c^{\text{triplet-or-higher}}$ contains *only* terms of third degree or higher, i.e. is a term due to the sum of the interaction among there or more risk drivers (which is a clear financial meaning).

References

[1] Carl Bacon. *Practical Portfolio Performance Measurement and Attribution*. Wiley Finance, second edition, 2008. 2, 5, 9

[2] George Boole. *Treatise on the calculus of finite differences*. London, MacMillan, third edition, 1880. 13, 25

[3] Andrew Colin. *Fixed Income Attribution*. Wiley Finance, 2005. 3

[4] The QuantLib group. A free/open-source library for quantitative finance. www.quantlib.org. 2

[5] Ralf Hudert. A modern way of fixed-income attribution. *CFA Institute, Investment Performance Measurement Feature Articles*, January, 2011. 3

[6] Timothy J. Lord. The attribution of portfolio and index returns in fixed income. *Journal of Performance Measurement*, **60**(4), Fall 1997. 10

[7] Marco Marchioro. A risk decomposition framework consistent with performance measurements. In *Risk and Performance Attribution*, Quantitative Research Series. StatPro website, January 2017. Permanent link marchioro.org/papers/risk-measure-components/. 2, 9, 22

[8] Attilio Meucci. The prayer: Ten-step checklist for advanced risk and portfolio management. *GARP Risk Professional*, April/June, 2011. 22

[9] Heinrich Niederhausen. Finite operator calculus with applications to linear recursions. www.math.fau.edu/Niederhausen. 13, 14