Fixed-income performance contributions of complex bonds

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Abstract

We consider the problem of finding the single-period fixed-income performance contributions of a generic security with a complex cash-flow structure. We extend the traditional fixed-income contribution model, based on the duration and the convexity, to include securities with generic floating-rate coupons and possibly with embedded call, put, or convertibility options. Using this method we can also compute the contributions of amortizing bonds and mortgage-backed securities and, as a special case, we also handle interest-rate only derivatives such as interest-rate futures. In this formulation we compute carry contributions exactly, i.e. without resorting to the traditional approach that the carry is proportional to the yield to maturity. Also in order to properly deal with the effect of the curve slope we define, for each security, a correction to the interest rate that summarizes the non-flatness effects. Finally and most importantly, we show how to compute the contributions of market-traded bonds with respect to their quoted price improving on the traditional approximation that is simply based on the model price.

Keywords: performance contribution, fixed-income attribution, bond pricing, interest-rate derivatives, fixed-income securities, pricing functions, interest-rate contribution, credit contribution, cash-flow discounting, spread/rate contribution, carry contribution, convergence contribution, coupon contribution

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1 Computation of contributions for LIBOR-indexed bonds

The performance measurement of a portfolio is a relatively mature subject in modern asset-management theory (see, e.g., reference [1]). Indeed the standard techniques allow us to split the portfolio return into bits that can be attributed to each of its composing securities. Therefore in any given evaluation period we are able to compute what is the contribution to the portfolio performance of each financial product. For fixed-income assets, however, a single number for each security is often not sufficient to describe the return, since interest-rate changes and credit-spread variations can affect many securities at the same time. Indeed while equities are only loosely correlated to their index, the bond-price movements are often tightly linked to the displacements of a number of interest rates and credit spreads. As a consequence bond portfolio managers prefer to further split the performance of each security into different fixed-income contributions and aggregate them together at portfolio level. We use the term *fixed-income contribution*, or simply *contribution*, for each of the performance components in which a security return can be split into.

The goal of the present paper is to define a contribution model that can be used for a very large class of fixed-income securities. However, in order to simplify the method illustration, we focus on the performance of an individual instrument during a single period, which is typically taken as one day. The aggregation to multiple periods and the computation of the portfolio contributions from the single-asset one, can be worked out using the standard performance-measurement tools available in literature (see for example reference [1]) and is not discussed in the present paper.

We ease into the discussion of the method for the generic security by defining a contribution model for the specific case of LIBOR-indexed corporate bonds. Later we extend the model to encompass securities with a more generic cash-flow structure.

1.1 Evaluation of a LIBOR-indexed corporate bond

Consider a LIBOR-indexed coupon bond issued by a corporate entity. The three main variables associated with the bond price are: the *current time* t, the *interest rates* and the *credit spread*. While the first variable, the current time, is deterministic the others, the interest rates and the credit spread, are market dependent and their evolution could be better described by stochastic processes. The bond price depends on the current time because coupons are paid at specific maturity dates, hence a variation in the time lags between the evaluation date and the coupon maturities results in a change of the bond present value. The interest-rate dependence arises both because rates affect the discount factors at future coupon dates and because some cash flows may be rate-dependent (as is usually the case for LIBOR-indexed notes). The dependence on the credit spread summarizes the market expectations of the probability of future default events (which also include restructuring events).

Even though both the dependence from the inter-

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est rates and that from the credit spread vary with the cash-flow maturity, we consider here the simpler case of a maturity-independent credit spread s and a flat term structures uniquely specified by a single interest rate r. More precisely we assume that the given bond will pay, at future dates $T_1, \ldots, T_n$, a number n of cash flows $C_1, \ldots, C_n$, given by,

$$C_i = A_i + B_i\, F_i(r)\,, \tag{1}$$

where $A_1, B_1, \ldots, A_n, B_n$, are given constants[1] and the coefficients $F_i(r)$'s are the risk-neutral estimates of the coupon LIBOR rates with i=1, ..., n. We also assume that all payments of principals are already included into the terms A_i's.

In this simplified model we presume the LIBOR discount factor at a maturity T_i to be determined by a flat interest rate r, with continuous compounding, as

$$L(T_i) = e^{-r\,(T_i - t)}\,, \tag{2}$$

where t the bond settlement date. (Since we are in the context of performance the evaluation date, which may be earlier then the settlement date, is irrelevant.) As shown for example in reference [2], the arbitrage-free LIBOR rate $F_i(r)$, for coupon i, can be computed from the discount curve as

$$\begin{aligned} F_i(r) &= \frac{1}{T_i - T_{i-1}}\left[\frac{L(T_{i-1})}{L(T_i)} - 1\right] \\ &= \frac{1}{T_i - T_{i-1}}\left[\frac{e^{-r\,T_{i-1}}}{e^{-r\,T_i}} - 1\right] \\ &= \frac{e^{r\,(T_i - T_{i-1})} - 1}{T_i - T_{i-1}}\,. \end{aligned} \tag{3}$$

In order to model the market expectations of future defaults we assume a flat credit spread s. We define the credit-aware discount factor $D(T_i)$ by adding s to the interest rate r so that we compute the discount factor as

$$D(T_i) = e^{-(r+s)(T_i - t)} = L(T_i)e^{-s\,(T_i - t)}\,.$$

More generally, given a maturity-dependent interest-rate curve $L(T)$ we define the discount at maturity T_i as

$$D(T_i) = L(T_i)e^{-s\,(T_i - t)}\,. \tag{4}$$

We can then compute the bond dirty price P by summing up the discounted the expected future cash flows:

$$\begin{aligned} P(r,s,t) \;=\; & \big[A_1 + B_1\, F_i(r)\big]\, D(T_1) + \ldots \\ & + \big[A_n + B_n\, F_n(r)\big]\, D(T_n)\,. \end{aligned}$$

Substituting into this equation the expression for the discount factor $D(T)$, as defined by equation (4), we obtain a more explicit formula for the bond pricing function:

$$\begin{aligned} & P(r,s,t) = \\ & \quad [A_1 + B_1\, F_1(r)]\, e^{-(r+s)(T_1 - t)} + \ldots \\ & \quad\quad + [A_n + B_n\, F_n(r)]\, e^{-(r+s)(T_n - t)}\,. \end{aligned} \tag{5}$$

Furthermore, by substituting the expression for the forward rates F_i's given by equation (3), we obtain an explicit formula for the bond dirty price that can be used to evaluate market-traded bonds at the settlement date.

1.2 Generic pricing functions and the carry term

In equation (5) we provide a useful formula to compute the bond settlement value from the interest rate r and the credit spread s. We derived this expression for the special case of LIBOR-index bonds and, in general, other securities will have different formulas still function of the variables t, r, and s (and possibly other risk drivers). We now consider a generic security for which we can compute the dirty price P_0 at date t_0 using a certain pricing function P, i.e. we can write

$$P_0 = P(r_0, s_0, t_0)\,.$$

Suppose that t_0 is the beginning of a period for which we want to compute the security performance and that t_1 denotes the end of that period. Since the bond pricing function does not change from t_0 to t_1 we have

$$P_1 = P(r_1, s_1, t_1)\,,$$

[1] The coefficients A_i's and B_i's already include the coupon year fraction, also B_i=0 when the i-th LIBOR rate has been fixed.

and we can write the relative return as

$$\mathrm{Ret} = \frac{P_1 + C_1 - P_0}{P_0}\,,$$

where C_1 is the sum of the coupons paid after t_0, however on or before t_1. The goal of this paper is to split the above return into contributions that can be assigned to the passage of time, to the change of interest rates and to the variation of the credit spread.

The carry contribution The easiest performance contribution that we can compute is that given by the passage of time. Indeed we can write the relative return as

$$\begin{aligned}\mathrm{Ret} &= \frac{1}{P_0}\left[P(r_1,s_1,t_1) + C_1 - P(r_0,s_0,t_0)\right]\\ &= \frac{1}{P_0}\left[P(r_1,s_1,t_1) - P(r_0,s_0,t_1)\right.\\ &\qquad \left. + P(r_0,s_0,t_1) + C_1 - P(r_0,s_0,t_0)\right]\\ &= C^{\mathrm{driv}} + C^{\mathrm{carry}}\,, \end{aligned} \qquad (6)$$

where the risk-driver contribution C^{driv} is computed by evaluating the pricing function at t_1 and the carry term C^{carry} is defined as

$$C^{\mathrm{carry}} = \frac{P(r_0,s_0,t_1) + C_1 - P(r_0,s_0,t_0)}{P_0}\,; \qquad (7)$$

we note that in this expression there is *no evolution* of the credit spread and the interest rate from t_0 to t_1. On the other hand the whole risk-driver *variation* is embedded in the risk-driver contribution C^{driv}, defined as

$$C^{\mathrm{driv}} = \frac{P(r_1,s_1,t_1) - P(r_0,s_0,t_1)}{P_0}\,, \qquad (8)$$

that is strictly evaluated at the end of the period t_1.

Note that, in order to compare the exact contribution C^{carry} with the traditional carry computed as $Y\cdot\Delta t$, where Y is the yield to maturity, it is convenient to write

$$C^{\mathrm{carry}} = T_D\cdot\Delta t\,, \qquad (9)$$

with $\Delta t{=}t_1 - t_0$ and the carry sensitivity T_D defined as

$$T_D = \frac{1}{P_0}\,\frac{P(r_0,s_0,t_1) + C_1 - P(r_0,s_0,t_0)}{\Delta t}\,. \qquad (10)$$

1.3 Expansion of the risk-driver contribution

In order to evaluate the risk driver contribution C^{driv} defined in equation (8), we perform the second-order Taylor expansion of the bond pricing function around the values s_1 and r_1:

$$\begin{aligned} P_1(r_1,s_1) - P(r_0,s_0) \simeq \frac{\partial P_1}{\partial s}\Delta s + \frac{\partial P_1}{\partial r}\Delta r \qquad (11)\\ -\frac{\partial^2 P_1}{\partial r^2}\frac{\Delta r^2}{2} - \frac{\partial^2 P_1}{\partial s^2}\frac{\Delta s^2}{2} - \frac{\partial^2 P_1}{\partial r\partial s}\cdot\Delta r\cdot\Delta s\,, \end{aligned}$$

where we neglect cubic and higher-order terms in Δr and Δs, and we define

$$\Delta s = s_1 - s_0 \qquad \text{and} \qquad \Delta r = r_1 - r_0\,.$$

In the above approximation (11) we use the notation P_1 for the bond pricing function to stress that the partial derivatives are computed at $t{=}t_1$. The expansion above has one main differences with respect to the traditional computation of fixed-income performance contributions: we compute the derivatives at $t{=}t_1$, $r{=}r_1$, and $s{=}s_1$ while traditionally they are computed $t{=}t_0$, $r{=}r_0$, and $s{=}s_0$. This choice accounts for the difference in the form of the fixed-income contributions compared to more traditional approaches and allows for smaller residuals in the carry term.

With the help of the approximate equation (11) we can write the risk-driver contribution C^{driv} as the sum of six components:

$$C^{\mathrm{driv}} = C^s_{1\mathrm{st}} + C^r_{1\mathrm{st}} + C^s_{2\mathrm{nd}} + C^r_{2\mathrm{nd}} + C^{rs}_{2\mathrm{nd}} + \varepsilon\,, \qquad (12)$$

where ε denotes the residual and the contributions C^d_{order}'s are computed below. Expression (12) is the base for the computation of the fixed-income contributions in this paper.

The credit-spread sensitivity According to equation (5) we can compute the pricing function spread derivative as

$$\frac{\partial P_1}{\partial s} = -\sum_{i=1}^{n}(T_i - t_1)\left[A_i + B_i\,F_i(r)\right]\cdot e^{-(r+s)(T_i-t_1)}\,.$$

Since the sum of the terms on the right-hand side divided by the bond price roughly equals the average of

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the (negative) time to maturity for each coupon, it is customary to define the *spread duration* SD_1 as,

$$SD_1 = -\frac{1}{P_1}\frac{\partial P_1}{\partial s}\,.$$

The spread duration is typically expressed in years and provides the average time scale at which variations in the credit spread are propagated to the bond price. Using the above definition of spread duration we can write the first-order spread contribution as

$$C^s_{1^{\rm st}} = -\gamma_{01}\cdot SD_1\cdot \Delta s\,, \tag{13}$$

where the rescaling factor γ_{01}, defined as

$$\gamma_{01} = \frac{P_1}{P_0}\,, \tag{14}$$

is needed because the return is relative to P_0 while the spread duration at t_1 is normalized with respect to P_1.

The interest-rate sensitivity Similarly to the spread sensitivity we can compute the pricing-function derivative with respect to the interest rate and define the *interest-rate duration* LD_1 as

$$LD_1 = -\frac{1}{P_1}\frac{\partial P_1}{\partial r}\,.$$

Explicitly computing the derivative from equation (5) we obtain

$$\begin{aligned}\frac{\partial P_1}{\partial r} = & -\sum_{i=1}^{n}(T_i - t_1)\,[A_i + B_i\,F_i(r)]\,\cdot \\ & \cdot e^{-(r+s)(T_i-t_1)} \\ & +\sum_{i=1}^{n} B_i\,F_i'(r)\,e^{-(r+s)(T_i-t_1)}\,,\end{aligned} \tag{15}$$

where, for each $i = 1,\dots,n$, $F_i'(r)$ is the derivative of the forward rate with respect to r, i.e.

$$F_i'(r) = e^{r\,(T_i-T_{i-1})}\,.$$

In equation (15) we notice the first group of terms on the right-hand side, i.e. the terms in the first and second line, to be exactly equal to the corresponding terms in the spread derivative. Hence, we define the second group of terms to be the *floating-rate extra term* f, i.e.

$$f_1(r) = \sum_{i=1}^{n} B_i\cdot e^{r\,(T_i-T_{i-1})}\cdot e^{-(r+s)(T_i-t_1)}\,.$$

so that, as a consequence of the previous equations, we have

$$LD_1 = SD_1 - \frac{f_1}{P_1}\,.$$

Notice that for fixed-rate coupon bonds all terms B_i's are null and the floating-rate extra term f is null, so that in this special case and we have $LD_1{=}SD_1$.

Using the above definition of the LIBOR duration we can write the first-order interest-rate contribution $C^r_{1^{\rm st}}$ as

$$C^r_{1^{\rm st}} = -\gamma_{01}\cdot LD_1\cdot\Delta r\,. \tag{16}$$

where the factor γ_{01} is defined as in equation (14).

Second-order sensitivities The last three terms on the right-hand side of equation (11) are the second-order terms of the pricing-function Taylor expansion. We define the second-order sensitivities, also known as convexities, as

$$CV_1^r = \frac{1}{P_1}\frac{\partial^2 P_1}{\partial r^2}\,,\qquad CV_1^s = \frac{1}{P_1}\frac{\partial^2 P_1}{\partial s^2}\,,$$

and

$$CV_1^{rs} = \frac{1}{P_1}\frac{\partial^2 P_1}{\partial r\partial s}\,.$$

From equation (5), it is easy to show that, for example, the spread convexity CV_1^s can be computed as

$$\begin{aligned}P_1\cdot CV_1^s = & \sum_{i=1}^{n}(T_i - t_1)^2\cdot[A_i + B_i\,F_i(r)]\,\cdot \\ & \cdot e^{-(r+s)(T_i-t_1)}\,,\end{aligned}$$

and similarly for the other second-order sensitivities.

Using the above definitions for the convexities the second-order contributions of equation (12) can be written as

$$C^s_{2^{\rm nd}} = -\frac{\gamma_{01}}{2}\cdot CV_1^s\cdot(\Delta s)^2\,, \tag{17}$$

$$C^r_{2^{\rm nd}} = -\frac{\gamma_{01}}{2}\cdot CV_1^r\cdot(\Delta r)^2\,, \tag{18}$$

$$C^{rs}_{2^{\rm nd}} = -\gamma_{01}\cdot CV_1^{rs}\cdot\Delta s\cdot\Delta r\,, \tag{19}$$

where, again, the rescaling factor γ_{01} is defined as in equation (14).

1.4 Generic two-factor contributions

In the previous subsection we explicitly provide an analytical form for the contributions C^d_{order}'s in the performance decomposition (12) for the LIBOR bond described by the pricing function (5). Substituting all terms in expression (12) into equation (6) we obtain the following performance split into contributions:

$$\begin{aligned} \text{Ret} &= T_D\,\Delta t + C^s_{1\text{st}} + C^r_{1\text{st}} \\ &\quad + C^s_{2\text{nd}} + C^r_{2\text{nd}} + C^{rs}_{2\text{nd}} + \varepsilon\,. \end{aligned} \tag{20}$$

This equation summarizes the fixed-income contribution model for a LIBOR-indexed corporate bond with mixed fixed-rate/floating-rate payments. There are seven contributions:

1. the carry term $T_D \cdot \Delta t$,
2. the first-order spread term $C^s_{1\text{st}}$ proportional to $SD_1 \cdot \Delta s$,
3. the first-order interest-rate term $C^r_{1\text{st}}$ proportional to $LD_1 \cdot \Delta r$;
4. three convexity terms: one for the spread ($C^s_{2\text{nd}}$), one for the interest-rate ($C^r_{2\text{nd}}$), and one with contributions from both interest rates and credit spread ($C^{rs}_{2\text{nd}}$, which however should be allocated to the spread contributions);
5. finally we also have the residual performance ε not accounted in the other terms.

Once the return components are computed for all securities in a portfolio they can be combined to form the portfolio contributions. We leave the details of the aggregation from single assets to the portfolio contributions to the references in the bibliography.

In the following sections we generalize the definitions the interest-rate duration LD, the spread duration SD, and the convexities, so that equation (20) can be applied to a much wider range of financial instruments and not only to the bonds with cash flows given by expression (1).

Fixed-rate bond model So far we considered the generic case of a LIBOR-indexed bond. However the generic expression for the cash flow used, see equation (1), can be also employed to compute the performance contributions in the specific case of fixed-rate coupon bonds simply by setting all coefficients B_i's to zero. As already noted earlier, in this case we have a null floating-rate extra term f_1 so that $SD_1{=}LD_1$. It is also easy to prove that we also have $CV^r_1{=}CV^s_1{=}CV^{rs}_1$. Since for fixed-rate bonds the spread duration and the interest-rate duration are identical it is customary to define a single sensitivity, called the *modified duration* and denote it as MD_1, as

$$MD_1 = SD_1 = LD_1\,. \tag{21}$$

Also, since for fixed-rate coupon bonds we have $CV^s_1{=}CV^r_1{=}CV^{rs}_1$ we can simplify equation (20) to become

$$\text{Ret} = T_D\,\Delta t - \gamma_{01}\,MD_1\cdot\Delta y - \gamma_{01}\frac{1}{2}CV_1\,(\Delta y)^2 + \varepsilon\,, \tag{22}$$

where $CV_1{=}CV^r_1$ and the delta yield Δy is defined as

$$\Delta y = \Delta r + \Delta s\,.$$

In summary, for fixed-rate coupon bonds, the interest-rate and the credit-spread terms can be combined into a yield equal to the sum of the interest rate and the credit spread.

1.5 Computation of sensitivities with respect to the clean price

The fixed-income contribution model described so far is based on the sensitivity computations with respect to the dirty price. This is the most generic case since not all securities pay coupons. However, for bonds that do pay a coupon it is customary to express the sensitivities in term of the clean price. In this subsection we provide the correction factors needed when the sensitivities are computed with respect to the clean price.

Given a generic pricing function $P(r, s, t)$ for the bond dirty price, as a practical example one can think at equation (5), for a security that pays coupon we

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can define the accrual amount $A(t)$ as the periodically increasing function that resets at each coupon date (recall that t is the bond settlement date). The accrual function specifies what is the part of coupon that should be paid to the current bond holder. The bond clean price $\hat{P}(r,s,t)$ is defined as

$$\hat{P}(r,s,t) = P(r,s,t) - A(t)\,. \tag{23}$$

Note that $A(t)$, since it is deterministic once the coupon has been established, does not depend on the interest rate nor on the credit spread.

We can invert this equation to compute the bond dirty price from the clean price as

$$P(r,s,t) = \hat{P}(r,s,t) + A(t)\,. \tag{24}$$

Since the accrual term does not depend on either r or s, the partial derivatives of P can be evaluated in terms of the corresponding derivatives of $\hat{P}$:

$$\frac{\partial P}{\partial s} = \frac{\partial \hat{P}}{\partial s}\,, \qquad \frac{\partial P}{\partial r} = \frac{\partial \hat{P}}{\partial r}\,,$$
$$\frac{\partial^2 P}{\partial r^2} = \frac{\partial^2 \hat{P}}{\partial r^2}\,, \qquad \frac{\partial^2 P}{\partial s^2} = \frac{\partial^2 \hat{P}}{\partial s^2}\,,$$

and

$$\frac{\partial^2 P}{\partial r \partial s} = \frac{\partial^2 \hat{P}}{\partial r \partial s}\,.$$

These expressions can then be used to compute the dirty-price sensitivities in terms of the clean-price ones:

$$SD = -\frac{1}{P}\frac{\partial P}{\partial s} = \hat{\gamma} \cdot \hat{SD}\,,$$
$$LD = -\frac{1}{P}\frac{\partial P}{\partial r} = \hat{\gamma} \cdot \hat{LD}\,,$$

for the first-order sensitivities and

$$CV^r = \frac{1}{P}\frac{\partial^2 P}{\partial r^2} = \hat{\gamma} \cdot \hat{CV}^r\,,$$
$$CV^s = \frac{1}{P}\frac{\partial^2 P}{\partial s^2} = \hat{\gamma} \cdot \hat{CV}^s\,,$$
$$CV^r = \frac{1}{P}\frac{\partial^2 P}{\partial r \partial s} = \hat{\gamma} \cdot \hat{CV}^r\,,$$

for the second-order sensitivities, where the clean/dirty *price ratio* $\hat{\gamma}$ is defined as

$$\hat{\gamma} = \frac{\hat{P}}{P}\,, \tag{25}$$

i.e. the ratio between the clean price and the dirty price.

Carry term as convergence plus coupon From definition (10) of T_D and expression (24) for the dirty price we have

$$\begin{aligned} T_D &= \frac{1}{P_0}\frac{\hat{P}(r_0,s_0,t_1) - \hat{P}(r_0,s_0,t_0)}{\Delta t} \\ &\qquad + \frac{1}{P_0}\frac{C_1 + A(t_1) - A(t_0)}{\Delta t} \\ &= \hat{\gamma}_0 \left(\hat{T}_D + \hat{C}_D\right)\,, \end{aligned} \tag{26}$$

where

$$\hat{T}_D = \frac{1}{\hat{P}_0}\frac{\hat{P}(r_0,s_0,t_1) - \hat{P}(r_0,s_0,t_0)}{\Delta t}\,, \tag{27}$$

is the carry sensitivity computed in terms of the clean price and $\hat{\gamma}_0$ is the clean/dirty price ratio at $t{=}t_0$; furthermore

$$\hat{C}_D = \frac{1}{\hat{P}_0}\frac{C_1 + A(t_1) - A(t_0)}{\Delta t}\,, \tag{28}$$

is the coupon sensitivity relative to the clean price. The above equation (26) provides a natural split of the carry sensitivity in two parts: a *convergence contribution*, proportional to $\hat{T}_D$, and a *coupon contribution* proportional to $\hat{T}_D$.

Example: fixed-rate bond For example, consider a fixed-rate coupon bond that has d settlement days, assuming the next coupon to be equal to the fixed amount C_1 and that we are far from the coupon date, the accrual can be computed as

$$A(t) = c^{\%}\, F_A\,(t - T_0)\,, \tag{29}$$

where the coupon C_1 is expressed as a percentage $c^{\%}$ of the face amount F_A, i.e.

$$C_1 = c^{\%}\,(T_1 - T_0)\,F_A\,. \tag{30}$$

Expression (28) for $\hat{C}_D$ can then be written as

$$\begin{aligned} \hat{C}_D &= \frac{c^{\%}\,F_A}{\hat{P}_0}\frac{(t_1 - T_0) - (t_0 - T_0)}{\Delta t} \\ &= \frac{c^{\%}\,F_A}{\hat{P}_0}\,. \end{aligned} \tag{31}$$

Fixed-income return contributions Gathering all terms together, with the carry term split into two parts, we can rewrite the fixed-income contribution model (20) as

$$\frac{P_1 - P_0}{P_0} = T_{\text{cnv}} \cdot \Delta t + T_{\text{cpn}} \cdot \Delta t + C^s_{\text{1st}} + C^r_{\text{1st}} + C^s_{\text{2nd}} + C^r_{\text{2nd}} + C^{rs}_{\text{2nd}} + \varepsilon\,, \qquad (32)$$

where

$$T_{\text{cnv}} = \hat{\gamma}_0 \cdot \hat{T}_D \qquad \text{and} \qquad T_{\text{cpn}} = \hat{\gamma}_0 \cdot \hat{C}_D\,.$$

Equation (32) splits the single-period return into seven terms and a residual.

2 Sensitivities for simple bonds

In the previous section we define a fixed-income contribution model for a generic bond depending on an interest rate r and a credit spread s. As already mentioned earlier, in the general case we need to resort to a numerical approximation to compute the bond sensitivities. However, there is a category of bonds that we denote as *simple bonds* for which it is possible to compute the sensitivities analytically. We dedicate this section to those bonds and we show how to compute their sensitivities. We call *simple bonds* either zero-coupon bonds or fixed-rate coupon bonds as defined later in this section. Since, as shown earlier, simple bonds depend from the interest rates and the credit spreads jointly as the yield, in this section we focus on the bond yield y as the independent variable.

2.1 Definitions of simple bonds

Simple bonds can either be zero-coupon bonds or fixed-rate coupon bonds. In the following paragraphs we describe both categories.

Zero-coupon bonds A zero-coupon bond is a fixed-income instrument described by a single date and returns the notional amount at this date. Given a maturity date T and assuming the notional amount to be one unit of currency, the present value $Z(y, T)$ of a zero-coupon bond can be expressed as a function of a yield y depending on the compounding convention. For simple compounding we have

$$Z_s(y, T) = \frac{1}{1 + y\,T}\,, \qquad (33)$$

for period compounding,

$$Z_k(y, T) = \frac{1}{\left(1 + \frac{y}{k}\right)^{k\,T}}\,, \qquad (34)$$

where k is the number of periods in a year. The special case of k=1 is usually named annual compounding:

$$Z_a(y, T) = \frac{1}{(1 + y)^T}\,. \qquad (35)$$

Finally, for continuous compounding we can write

$$Z_c(y, T) = e^{-y\,T}\,. \qquad (36)$$

It is a market custom to quote the bond yield using simple compounding for maturities less than a year and annual compounding for maturities greater than a year:

$$Z_m(y, T) = \begin{cases} Z_s(y, T) & \text{for } T \le 1 \\ Z_a(y, T) & \text{for } T > 1 \end{cases}\,. \qquad (37)$$

In the remainder of this paper, independently from the compounding convention used, we generically denote the zero-coupon bond price as $Z(y, T)$.

Note that simple compounding and annual compounding match for a maturity of exactly one year. Also, since

$$\lim_{k\to\infty} Z_k(y, T) = Z_c(y, T)\,,$$

continuous compounding is equivalent to period compounding over an infinite number of periods.

Flat interest-rate term structures In computing the yield of complex bonds it is useful to define *flat interest-rate term structures*. Hence, given a yield y and a compounding convention, we can define the discount factor $D^y(T)$ of a flat interest-rate term structure as

$$D^y(T) = Z(y, T)\,. \qquad (38)$$

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Since this term structure also provides the price of a zero-coupon bond with the same maturity the variable y is also known as the zero rate. In this paper we use the same symbol $Z(y,T)$ to denote both the discount factor of a flat interest-rate curve described by y and the price of a zero-coupon bond.

Fixed-rate coupon bonds We define a fixed-rate coupon bond as a security paying n known coupons C_1, C_2, ..., C_n, at known future dates T_1, T_2, ..., T_n, and a redemption R_n at T_n. Given a yield y and a compounding convention we can compute the coupon bond pricing function $P(y,t)$ at the current time t on a flat term structure as

$$\begin{aligned} P(y) \;=\; & Z(y, T_1 - t)\, C_1 + \dots \\ & + Z(y, T_n - t)\, C_n + Z(y, T_n - t)\, R_n \,. \end{aligned} \tag{39}$$

Similarly to the zero-coupon bond described earlier, the bond pricing function establishes a link, usually a unique map, between the bond price and the bond yield. Hence, a quoted bond with a price P^q uniquely determines a bond yield y^q satisfying

$$P^q = P(y^q)\,.$$

The interest rate y^q is usually called the bond *internal rate of return*, or the *bond yield*, corresponding to quoted price P^q.

2.2 First-order sensitivity: the modified duration

We now define the concept of the first-order yield sensitivity for zero-coupon bonds and fixed-rate coupon bonds. Since bond traders usually think in terms of yield, price variations are translated into yield variations and vice versa. We use the mathematical derivatives of the bond pricing function to determine what is the relative bond-price variation for a small change in the bond yield. For a bond with price $P(y)$ we define the modified duration MD as

$$MD = -\frac{1}{P}\frac{\partial P}{\partial y}\,. \tag{40}$$

Notice that the modified duration has the same unit of measure as time. Expression (40) can be inverted to determine the yield derivative of price in terms of the price itself and the modified duration:

$$\frac{\partial P}{\partial y} = -P \cdot MD\,. \tag{41}$$

Modified duration for zero-coupon bonds The modified duration for zero coupon bonds,

$$MD^z(y,T) = -\frac{1}{Z(y,T)}\frac{\partial Z(y,T)}{\partial y}\,, \tag{42}$$

can be computed analytically for each compounding convention. For simple compounding we have

$$\begin{aligned} MD^z_s(y,T) \;&=\; -\,(1+y\,T)\,\frac{(-T)}{(1+y\,T)^2} \\ &=\; \frac{T}{1+y\,T}\,, \end{aligned}$$

for period compounding,

$$\begin{aligned} & MD^z_k(y,T) \\ & \quad = -\left(1+\frac{y}{k}\right)^{k\,T}\left(\frac{-k\,T}{k}\right)\left(1+\frac{y}{k}\right)^{-k\,T-1} \\ & \quad = \frac{T}{1+\frac{y}{k}}\,, \end{aligned}$$

and, finally, for continuous compounding,

$$MD^z_c(y,T) = -e^{y\,T}\,(-T)\,e^{-y\,T} = T\,.$$

In the special case of annual compounding the modified duration can be computed as

$$MD^z_a(y,T) = \frac{T}{1+y}\,,$$

that can also be used to compute the modified duration for market compounding.

Note that, to the first order in y, for each compounding convention we have

$$MD_c(y,T) \simeq T\,.$$

Therefore for zero-coupon bonds, the modified duration is roughly equal to the bond time to maturity.

Modified duration for fixed-rate coupon bonds We can compute the modified duration of a fixed-rate coupon bonds by taking the derivative of equation (39):

$$\begin{aligned} MD &= \frac{1}{P}\left[Z(y,T_1)\, MD^z(y,T_1)\, C_1 + \ldots + \right. \\ &\qquad \left. + Z(y,T_n)\, MD^z(y,T_n)\,(C_n + R_n)\right] \\ &= \frac{P^{MD}}{P}\,, \end{aligned} \tag{43}$$

where P^{MD} is the price of a bond paying coupons C_i^{MD} defined as

$$C_i^{MD} = MD^z(y,T_i)\, C_i \qquad \text{for } i = 1,\ldots,n\,,$$

and a redemption

$$R_n^{MD} = MD^z(y,T_n)\, R_n\,.$$

The modified duration of a fixed-rate coupon bond can be computed, according to equation (43) as the weighted average of the modified duration of the single coupons.

2.3 Second-order sensitivity: the convexity

As seen in the first section, sometimes it is necessary to approximate the bond pricing function beyond the first order in the yield. Hence we compute the yield second derivative of bond price and define the bond convexity CV as

$$CV = \frac{1}{P}\frac{\partial^2 P}{\partial y^2}\,. \tag{44}$$

We can invert this equation to derive the bond-price second derivative in terms of the convexity and the price:

$$\frac{\partial^2 P}{\partial y^2} = P \cdot CV\,. \tag{45}$$

The price convexity can also be expressed in terms of the modified duration and its yield derivative. From equation (41) we obtain,

$$CV = -\frac{1}{P}\frac{\mathrm{d}}{\mathrm{d}y}\left[P\cdot MD\right] = -\frac{MD}{P}\frac{\mathrm{d}P}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}y}MD\,,$$

from which we have,

$$CV = MD^2 - \frac{\mathrm{d}MD}{\mathrm{d}y}\,. \tag{46}$$

Since in most cases the second term is small, this equation states that the order of magnitude of the convexity is approximately proportional to the square of the modified duration.

Computation of convexity for zero-coupon bonds We can compute analytically the second derivative of the bond price with respect to yield for zero-coupon bonds for the different compounding conventions. For simple compounding we have

$$CV_s^z(y,T) = \frac{2\,T^2}{(1+y\,T)^2}\,,$$

for period compounding,

$$CV_k^z(y,T) = \frac{T\,(1+k\,T)}{k\,\left(1+\frac{y}{k}\right)^2}\,,$$

and, finally, for continuous compounding

$$CV_c^z(y,T) = T^2\,.$$

The special case of annual compounding is given by

$$CV_a^z(y,T) = \frac{T\,(1+T)}{(1+y)^2}\,,$$

that can also be used to compute the convexity for market compounding.

Convexity of coupon bonds We can compute analytically also the convexity of a fixed-rate coupon bond with the help of equation (46). To compute the convexity we only need to work out the yield derivative of the coupon-bond modified duration:

$$\begin{aligned} -\frac{\partial}{\partial y}MD &= -\frac{\partial}{\partial y}\left(\frac{P^{MD}}{P}\right) \\ &= \frac{P^{MD}}{P^2}\frac{\partial P}{\partial y} - \frac{1}{P}\frac{\partial P^{MD}}{\partial y}\,. \end{aligned}$$

Since, for each i,

$$\frac{\partial}{\partial y}\left[Z(y,T_i)\, MD^z(y,T_i)\right] = -Z(y,T_i)\, CV^z(y,T_i)\,,$$

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we have

$$\begin{aligned}\frac{\partial P^{MD}}{\partial y} &= -\left[Z(y,T_1)\,CV^z(y,T_1)\,C_1 + \ldots\right.\\ &\quad \left. + Z(y,T_n)\,CV^z(y,T_n)\,(C_n + R_n)\right]\\ &= -P^{CV}\,,\end{aligned}$$

where P^{CV} is the price of a coupon bond with coupons C_i^{CV}'s given by

$$C_i^{CV} = CV^z(y,T_i)\,C_i \quad \text{for } i = 1,\ldots,n\,,$$

and a redemption

$$R_n^{CV} = CV^z(y,T_n)\,R_n\,.$$

Bringing all the pieces together we can write the expression for the convexity of a coupon bond as,

$$CV = \frac{1}{P}\left(P^{CV} - MD\cdot P^{MD} + MD^2\cdot P\right)\,.$$

This expression, and the other results of this section, can be used to compute analytically the yield sensitivities of simple bonds. Hence for simple bonds we are able to analytically compute the contributions of the fixed-income contribution model describe by equation (22)

3 Sensitivities for complex bonds

The securities considered so far pay fixed or LIBOR-indexed coupons. Even though most bonds on the market posses these features, there is a large number of bonds that have a more complex structure. In this section we define a method that allows us to compute the sensitivities for a large class of complex bond.

The main difference between a coupon bond analyzed in the previous section and the more generic type of bonds considered here, is the presence of risk drivers other than the interest rates and the credit spread. Another important issue that we address in this section is how to handle an interest-rate term structure that is not flat.

Examples of complex bonds We provide here some examples of complex bonds. The list is not complete, however, it is representative of the most-popular instruments available on the markets.

- **Callable bonds**. A callable bond is a security bearing coupons, based upon a fixed or a floating rate, that the issuer has the right, at certain dates, to buy back at a pre-determined prices. These instruments are also known as bonds with embedded call options.

- **Puttable bonds** A puttable bond gives the holder the right to exchange his investment for cash at specific dates for a given redemption amount. Unlike callable bonds, where the issuer has the right to payback the instrument holder, in a puttable bond is the investor that decides whether to exercise the option. Hence it may happen that one investor decides to exercise her option while another may not. These instruments are also known as bonds with embedded put options.

- **Bonds with auto-callability features**. A bond, typically an equity-linked note, is said to have auto-callability features when it is redeemed by the issuer upon the market verifying certain conditions. For example, a convertible bond that is converted and paid out when the underlying stock price touches a barrier is an instrument with an auto-callability features.

- **Mortgage-backed securities** A mortgage-backed security (MBS for a short) is a bundle of mortgages, repacked and sold as a single bond. When one or more home owners, belonging to the mortgage pool, decide to payback their mortgages, the principal paid back is tunneled to the bond holder and the instrument notional is reduced accordingly.

In the following subsections we define the sensitivities for complex bonds so that we are able to use the fixed-income contribution model defined in subsection 1.4.

The LIBOR-curve zero rate As mentioned earlier one of the major problems of using the method describe in section 1 to compute the return contributions, is that in the actual financial markets there is an entire interest-rate term structure and not a single rate r. Hence how do we pick a single interest rate representative of the whole curve? The obvious choice is to use the zero rate at the bond maturity. Therefore given a security with a maturity date T_m we define the associated zero rate r^z as the interest rate that provides the same discount as that of the whole curve $L(T)$, i.e.

$$Z(r^z, T_m) = L(T_m)\,, \tag{47}$$

where Z is a flat discount term structure such as those described in section 2. While r^z should be considered to be of the correct order of magnitude, as we show in the reminder of this section, there are better choices for the interest rate that should be used in the computation of contributions.

3.1 Sensitivities for interest-rate only securities

There are many cases in which the security pricing function does not depend on the credit spread but only on the interest-rate term structure $L(T)$. In this category we can put, for example, some LIBOR-indexed interest-rate options or, as another example, interest-rate futures. We use the notation $L(T)$ for the interest-rate curve because very often $L(T)$ is the LIBOR curve. For simplicity we omit to indicate the time variable t since the following computations can be performed both at the beginning and at the end of the period.

Consider a generic security for which we can write the model price P^m as

$$P^m = P(L(T), \mathcal{M})\,, \tag{48}$$

where the pricing function P has an explicit dependence on all nodes of the interest-rate term structure $L(T)$ and we collectively denote the other instrument variables, or market data, as $\mathcal{M}$. Notice that the credit-spread is not a risk driver for these securities and it *does not* belong to $\mathcal{M}$.

While the instrument dependence from the interest-rate curve may be very complex, in order to define an appropriate interest rate we need to find a single number r that summarizes the effects of $L(T)$. Therefore we consider a flat term structure $Z(r, T)$ depending on a single rate r. The function that associates the rate r to the security price $P(Z(r, T), \mathcal{M}, t)$ depends now on a single variable, hence, we can solve for the flat rate r^m that satisfies

$$P^m = P(Z(r^m, T), \mathcal{M})\,. \tag{49}$$

The model interest-rate r^m is the flat rate that summarizes the whole term structure and is sufficient to compute the bond price P^m on a flat interest-rate term structure. We call *zero shift*, and use the symbol δ^z, the difference between the model rate r^m and the zero rate r^z, i.e.

$$\delta^z = r^m - r^z \qquad \Longleftrightarrow \qquad r^m = r^z + \delta^z\,. \tag{50}$$

The zero shift is the extra bit of interest rate that should be added to the zero rate so that we obtain a security price in line with the full curved interest-rate term structure.

Notice that when we are computing the performance contributions across the period from t_0 to t_1 we need to evaluate the zero shift both at the beginning and at the end of the period. In terms of notation we write equation (50) as

$$r_0^m = r_0^z + \delta_0^z \qquad \text{and} \qquad r_1^m = r_1^z + \delta_1^z\,, \tag{51}$$

and we denote with δ_i^z the zero shift at time t_i, with i=0,1.

Duration and convexity for interest-rate only securities Given a interest-rate security with a pricing function described by equation (48), we define its interest-rate duration and its convexity to be proportional to the first and the second derivatives, with respect to the interest rate, of the function

$$P^z(r) = P\left(Z(r, T), \mathcal{M}\right)\,. \tag{52}$$

Note that from equation (49) we have

$$P^m = P^z(r^m)\,.$$

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More precisely we define the interest-rate duration and the convexity as

$$LD = -\frac{1}{P^m}\frac{\partial P^z}{\partial r} \quad \text{and} \quad CV^r = \frac{1}{P^m}\frac{\partial^2 P^z}{\partial r^2}\,, \tag{53}$$

where the derivatives are evaluated for r=r^m.

Numerical computation of sensitivities For most pricing functions we do not have an analytic expression for the yield derivative. To numerically compute LD and CV^r we can resort to a finite-difference scheme, such as that described in reference [3]. Therefore, given a numerical bump b, typically b=0.1%, we compute the shocked prices P_+ and P_- as

$$P_+ = P^z(r^m + b, t)\,, \quad \text{and} \quad P_- = P^z(r^m - b, t)\,, \tag{54}$$

so that the interest-rate duration and the convexity, respectively, can be approximated by

$$LD \simeq -\frac{P_+ - P_-}{2\cdot b\cdot P^m}$$

and

$$CV^r \simeq \frac{P_+ + P_- - 2\,P_m}{b^2\cdot P^m}\,.$$

Interest-rate contribution Since the sensitivities with respect to the credit spread are all zero, using the sensitivities just defined and the results of section 1 we can split the bond return as in equation (20) with

$$\begin{aligned}
C^r_{1\text{st}} &= -\gamma_{01}\cdot LD\cdot \Delta r^m\,,\\
C^s_{1\text{st}} &= 0\,,\\
C^r_{2\text{nd}} &= -\frac{\gamma_{01}}{2}\cdot CV^r\cdot(\Delta r^m)^2\,,\\
C^s_{2\text{nd}} &= 0\,,\\
C^{rs}_{2\text{nd}} &= 0\,,
\end{aligned}$$

where the carry is defined in equation (10) and γ_{01} is defined in equation (14). More details on the procedure needed to compute the fixed-income contributions for a interest-rate only security can be found in section 5.

3.2 Sensitivities for risky securities

In practice there are only a few pricing functions that do not depend on the credit spread. Indeed in most cases a credit spread of some sort is needed to compute the correct bond model price. Hence, in this subsection we describe a method to compute the sensitivities for bonds that depend both on an interest-rate term structure $L(T)$ and a credit spread s. Again for simplicity we omit to indicate the time variable t since the following computations can be performed both at the beginning and at the end of the period.

Computation of the zero shift Even in the presence of a credit spread we can benefit from a definition of a zero shift similar to that computed for interest-rate only securities. Again in order to load the difference between a flat and a non-flat interest-rate term structure on a single rate r^m we define the flattened pricing function P^z as

$$P^z(r, s) = P\left(Z(r, T), s, \mathcal{M}\right)\,, \tag{55}$$

and solve for the interest rate r^m that satisfies

$$P^m = P^z(r^m, s^m)\,, \tag{56}$$

where s^m is the best credit spread available to the quantitative model at the time of pricing.

Similarly to the interest-rate case we define the zero shift δ^z as in equations (50). Again, when computing the performance contributions across a period from t_0 to t_1 we would need to compute two zero shifts as in equations (51).

In the previous equations the zero shift was computed using the dirty price. Since, as shown by equation (23), the difference between the dirty price and the clean price only depends on the time variable, r^m can also be computed using the clean price by solving equation

$$\hat{P}_m = \hat{P}\left(Z(r^m, T), s^m, \mathcal{M}\right)\,,$$

where $\hat{P}$ is the *clean* pricing function.

Sensitivities for risky complex bonds The flattened pricing function P^z allows us to compute the security sensitivities. Indeed we can define the spread duration and the corresponding convexities by computing the first and the second derivatives of this function with respect to the spread:

$$SD = -\frac{1}{P^m}\frac{\partial P^z}{\partial s}\,, \qquad CV^s = \frac{1}{P^m}\frac{\partial^2 P^z}{\partial s^2}\,, \tag{57}$$

and

$$CV^{rs} = \frac{1}{P^m}\frac{\partial^2 P^z}{\partial s \partial r}\,, \tag{58}$$

by evaluating the result for $s{=}s^m$, $r{=}r^m$. Similarly to the interest-rate only case we can use expressions (53) to evaluate LD and CV^r.

Finite-difference computation of sensitivities Since most pricing functions for complex bonds are not easily cast into an analytic form, we usually need to resort to numerical methods for the computations of sensitivities. The finite-difference computation of sensitivities for spread-based pricing functions can be easily defined from equations (57) and (58) by using a finite-difference schema.

We consider a numerical bump b, e.g. b=0.1%, and compute the spread bumped prices as

$$P_{s+} = P^z(r^m, s^m + b)\,,$$

and

$$P_{s-} = P^z(r^m, s^m - b)\,.$$

Then the approximate values of the spread duration and the convexity can be computed as,

$$SD = -\frac{P_{s+} - P_{s-}}{2 \cdot b \cdot P^m}$$

and

$$CV^s = \frac{P_{s+} + P_{s+} - 2 \cdot P^m}{b^2 \cdot P^m}\,.$$

Given the same numerical bump b, we compute the interest-rate bumped prices as

$$P_{r+} = P^z(r^m + b, s^m)\,,$$

and

$$P_{r-} = P^z(r^m - b, s^m)\,,$$

so that the interest-rate duration can be approximated as

$$LD = -\frac{P_{r+} - P_{r-}}{2 \cdot b \cdot P^m}\,.$$

The convexity given purely by the interest rates can be computed similarly to that due to the credit spread:

$$CV^r = \frac{P_{r+} + P_{r+} - 2 \cdot P^m}{b^2 \cdot P^m}\,.$$

As shown by reference [3], the cross convexity between interest rates and credit spreads can be computed as

$$CV^{rs} = \frac{P_{++} + P_{--} - P_{-+} + P_{+-}}{4 \cdot b^2 \cdot P^m}\,,$$

where

$$P_{\pm\pm} = P^z(r^m \pm b, s^m \pm b)\,,$$

with an obvious notation.

Computation of performance contributions Using the above numerical approximations for the sensitivities of the flattened pricing function and the method outlined in subsection 1.4, we can compute the fixed-income contributions that can be associated to the variations in the instrument model price as in equation (20) with

$$C^r_{1\text{st}} = -\gamma_{01} \cdot LD \cdot \Delta r^m \quad \text{and} \quad C^s_{1\text{st}} = -\gamma_{01} \cdot SD \cdot \Delta s^m$$

for the first-order contributions and

$$\begin{aligned} C^s_{2\text{nd}} &= -\frac{\gamma_{01}}{2} \cdot CV^s \cdot (\Delta s^m)^2\,, \\ C^r_{2\text{nd}} &= -\frac{\gamma_{01}}{2} \cdot CV^r \cdot (\Delta r^m)^2\,, \\ C^{rs}_{2\text{nd}} &= -\gamma_{01} \cdot CV^{rs} \cdot \Delta s^m \cdot \Delta r^m\,, \end{aligned}$$

for the second-order contributions. Recall that γ_{01} is defined in equation (14).

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3.3 Computational details for specific asset types

In the previous paragraphs we describe a method to compute the sensitivities of complex bonds both analytically and numerically so that we can use the fixed-income contribution models described in section 1. While the computations are quite generic and can be applied to most financial instruments, for certain specific assets we need to specify more details in order to achieve better results. For example, let us consider the time to expiration, denoted earlier as T_m, of a bond that can be used to determine the zero interest rate from an interest-rate term structure. While for a zero-coupon bond and a fixed-rate coupon bond we should always choose the legal bond expiration (i.e., T_n in the notation of this paper), for other bonds this may not be an optimal choice. For some complex bonds, such as those considered in the following paragraphs, it is more sensible to define an effective time to maturity in a bond-specific way.

Amortizing bonds An amortizing bond is very similar to a fixed-rate coupon bond, however, part of the principal is paid back with each coupon. Given a series of coupons c_1, ..., c_n, to be paid at dates T_1, ..., T_n, based on notionals N_1, ..., N_n, the amortizing-bond pricing function can be written as

$$P(D(T)) = D(t_1)\,(c_1\,N_1 + R_1) + \ldots \\ \ldots + D(t_n)\,(c_n\,N_n + R_n)\,, \qquad (59)$$

where we explicitly wrote only the dependence from the discount factor $D(T)$ and only the future coupons have been considered. Since coupons need to be paid on the outstanding redemption, the constraints between the notionals and the redemptions are given by

$$N_2 = N_1 - R_1\,, \quad \ldots\,, \quad N_n = N_{n-1} - R_{n-1}\,,$$

and

$$R_n = N_n\,.$$

In order to define an effective time to maturity we need to consider the per-payments of redemptions. Therefore we define the effective time to maturity as the average time of redemption payments weighed on the corresponding reimbursed values,

$$T_m = \frac{T_1\,R_1 + T_2\,R_2 + \ldots + T_n\,R_n}{R_1 + R_2 + \ldots + R_n}\,.$$

Notice that in the special case of a standard fixed-rate coupon bond we have $R_1 = 0, \ldots, R_{n-1} = 0$, so that $T_m = T_n$.

Auto-callable bonds Consider now a bond with auto-callability features as described at the beginning of this section. This type of bonds are typically priced numerically using Monte Carlo simulations. Hence, consider N_s paths p_j's so that the bond model price is given by

$$P^m = \frac{1}{N_s}\sum_{j=1}^{N_s} PV(p_j)\,,$$

where $PV(p_j)$ is the present value of path j. Denoting with $T(p_j)$ the expiration date of the bond in path j, we define the expected time to maturity to be the average of $T(p_j)$ over all the paths, i.e:

$$T_m = \frac{1}{N_s}\sum_{j=1}^{N_s} T(p_j)\,.$$

Since the time to maturity thus defined is the average, with respect to the risk neutral measure, of the time to maturity on all paths, it is the correct expected time to maturity.

Mortgage-backed securities Mortgage- backed securities, also known as MBS, contain features of bonds with put options, however, with pre-payment of redemption at each coupon. The estimation of a price for mortgage-backed securities is typically performed using complex quantitative models that often include evaluation on finite-difference grids and whatnot. Hence the definition of an effective time to maturity becomes challenging. In practice therefore we can just treat a mortgage-backed security as an amortizing bond and neglect all other complications. We define the time to maturity for a mortgage-backed security by neglecting the effect of prepayments due to changes in interest rates, however, keeping into account the natural turnover in the mortgage prepayments.

Inflation-linked bonds There are two different ways to quote inflation-linked bonds: in cash terms and in real terms. Historically bonds have been quoted in cash terms until the early 2000s, however, after that period they were all quoted in real terms. The real-term quoted price P_r and the cash-based price P_c satisfy

$$P_r = i_r \cdot P_c \,,$$

where the inflation ratio i_r is defined as

$$i_r = \frac{I(T_0)}{I(T_s)} \,,$$

with $I(T)$ is the appropriate inflation index, T_0 is the bond issue date, and T_s is the bond settlement date. All yield sensitivities, i.e. T_D, SD, LD, and the convexities, should be computed on the cash-based price. However, it is usually easier to compute the real-term sensitivities SD_r, LD_r, and so on. In this case, since at any given date i_r is a constant, we have

$$T_D = \frac{T_D^{\text{real}}}{i_r} \,, \qquad LD = \frac{LD^{\text{real}} r}{i_r} \,, \quad SD = \frac{SD^{\text{real}}}{i_r} \,,$$

and similarly for the convexities. These sensitivities can then be use to compute the fixed-income contributions for inflation-linked bonds.

Bonds with embedded call and put options Earlier in this section we have shown how to compute the yield and the sensitivities for a large class of complex bonds. Given its generality, that same technique can be used for coupon bonds with embedded call and put options. Recall that a bond of this type, unless the option is exercised, pays $C_1, C_2, \ldots, C_n$, coupons at dates $T_1, T_2, \ldots, T_n$, and a redemption R_n at the maturity date T_n. We assume that the call and the put options are of the standard type, thus excluding *poison-put* optionality, soft callabilities, and other exotic option types. For simplicity we also assume, as is usually the case, that the option exercise dates coincide with the coupon dates $T_1, T_2, \ldots, T_{n-1}$. We excluded from the list the final maturity date T_n for the obvious reason that at T_n, if no option was exercised earlier, the redemption is always paid back.

Before we outline how to define a time to maturity for these bond types we need some supplementary definitions. For each $j = 1, \ldots, n-1$, we denote, respectively, with R_j^c and R_j^p the redemptions when the call, respectively the put, option is exercised at date T_j. Since the call redemption must be larger than the corresponding put redemption we always have

$$R_j^c \geq R_j^p \qquad \text{for } j = 1, \ldots, n-1 \,.$$

For each coupon date T_j we define two *exercised coupon bonds*, one for the call option and one for the put option, that pay the same cash flows as the original bond, however that are always exercised at date T_j. Using the standard pricing function for fixed-rate coupon bonds we can compute the current price for the exercised bonds. We call P_j^c the bond price corresponding to the callability exercise at T_j, and P_j^p the bond price corresponding to the puttability exercise at the same date. More precisely, we can compute the price of each exercised bond using equation (39), i.e. for the call exercise we have

$$P_j^c(y) = Z(y, T_1)\, C_1 + \ldots + Z(y, T_j) \, \left[C_j \, + \, R_j^c\right] \,, \tag{60}$$

while for put exercise we can write

$$P_j^p(y) = Z(y, T_1)\, C_1 + \ldots + Z(y, T_j) \, \left[C_j \, + \, R_j^p\right] \,, \tag{61}$$

where, for short, we define the yield $y{=}r{+}s$.

Note that in these equations the prices were computed assuming the exercised bonds to have the same yield as the original callable bond. Finally we define the maturity bond pricing function $P_n(y)$ as the theoretical bond corresponding to the original callable bond, however without any callability or puttability options:

$$P_n(y) = Z(y, T_1)\, C_1 + \ldots + Z(y, T_n) \, [C_n \, + \, R_n] \,. \tag{62}$$

We first briefly describe in words the method to compute the effective maturity date and later provide a more precise mathematical description. Since the exercised bonds and the maturity bond are fixed-rate coupon bonds, their time to maturity is simply defined as the time at which their redemption is paid, i.e., $T_1, \ldots, T_n$. The time to maturity of the callable bond

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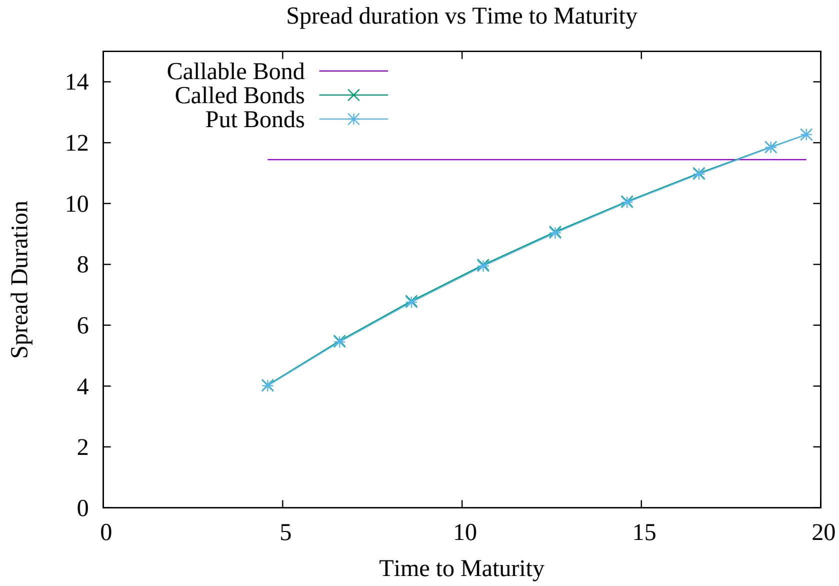

Figure 1: Plot of the callable-bond spread duration (straight flat line) and the computed spread durations for exercised call and put bonds. The considered callable bond has a legal maturity of 20 years and is both callable and puttable every two years starting from the fourth year. In this example the effective time to maturity turns out to be close to 18 years.

is unknown because we are not sure when, or if, any of the option will be exercised. However we do know the bond spread duration, computed using the techniques of subsection 3.2, and the spread duration of all exercised bonds. Hence, we define the *effective time to maturity* of the a bond with call and put options as *the time to maturity of the exercised bond that has a spread duration closest to that of the given bond itself*.

We now provide a more precise mathematical definition of the effective time to maturity. Let us denote with SD_j^c the spread duration of the j-th call-exercised bond, with SD_j^p that of the j-th put-exercised bond, and with SD_n that of the maturity bond. The spread durations should be computed with a yield that is consistent with the bond price we would compute if they were commonly traded bonds. Thus each exercised bond has, in general, a different yield from the others that in turn might be different from that of the original bond. Typically the spread durations thus computed are monotonic in the time to maturity T_j, however as we are about to see, this is not a necessary condition to define the effective time to maturity. Starting with the exercised callability spread durations, we define a function $f^c(T)$ of the maturity T as the piecewise-linear interpolation of points

$$(t_1, SD_1^c), \quad \ldots\,, \quad (t_{n-1}, SD_{n-1}^c), \quad (t_n, SD_n)\,.$$

In the graph of figure 1 this function is plotted with a green color. Similarly we define a function $f^p(T)$ using the put spread durations SD_j^p's and SD_n. The function $f^p(T)$ is plotted in blue in the graph of figure 1. Then we denote with SD^y the spread duration of the original bond (see the red horizontal line in figure 1). For no arbitrage reasons we should always have

$$SD_1^c \leq SD^y \leq SD_n$$

and

$$SD_1^p \leq SD^y \leq SD_n\,,$$

hence the equations

$$SD^y = f^c(T^c) \quad \text{and} \quad SD^y = f^c(T^p)$$

should always have at least one solutions each for T^c and T^p. However since the spread durations are computed numerically, a small error could bring SD^y outside the ranges (MD_1^c, MD_n) and (MD_1^p, MD_n), in which case we define

$$T^c = \begin{cases} t_1 & \text{if} \quad SD^y < SD_1^c \\ & \\ t_n & \text{if} \quad SD^y > SD_n \end{cases} ,$$

and similarly for T^p. Finally we define the effective time to maturity for the original bond T_m as

$$T_m = \max\left(T^c, T^p\right) .$$

Notice that the effective time to maturity for the bond is always between the time to next coupon T_1 and the final time to maturity T_n. In the graph of figure 1 the effective time to maturity is approximately equal to 18 years.

4 Quoted-price contributions

The fixed-income contribution model defined in section 1 splits the single-period return into components according to the pricing-function sensitivities. The derivation is based on the assumption that the quantitative model can be used to exactly compute the security price. However, more often than not, the bond *model price* does not match the quoted one. There are many reasons for this discrepancy, the most important are:

- the issuer credit spread was not measured accurately,
- liquidity issues make a bond cheaper or dearer,
- the interest-rate curve used is slightly out of sync with the current market data,
- the correct quantitative model for the bond is not available or too expensive to be developed,
- the bond value was computed on mid curves but prices are observed on bid (or ask) quotes,
- the timing of the model-price computation does not match that of the market quote observation.

We show in this section how to compute the contributions that arise from the difference between the quoted and the model prices.

Regardless of the actual reason for the imperfect value obtained from the pricing function, we blame the difference between the model price and the quoted one to an adjustment of the risk drivers (i.e. the pricing-function variables). As discussed in section 1, the pricing function of a bond typically depends on at least three variables: the current time, the interest rate, and the credit spread. Leaving aside the time variable that cannot be adjusted, for interest-rate only securities we can only modify the interest rate in order to match the quoted price. For risky securities, in principle, we could recalibrate the model price to obtain the quoted one by displacing either the interest rate or the credit spread. In practice, however, the main reason that makes the model price different from the quoted one is a mismatch of the credit spread. Hence in the following derivations we assume that for risky bonds the market quote is different from the model price only because of a shift in the credit spread.

Since we assume all adjustments to be dependent difference between the quote P^q and the model price P^m we need to create a dimensionless parameter that represents this difference. Hence, we define the *quoted price displacement* ΔP as

$$\Delta P = \frac{P^q - P^m}{P^m} = \frac{P^q}{P^m} - 1 , \tag{63}$$

i.e. the return of the quoted price with respect to the model one. For example if a bond has a quoted price of P^q=103.78 and a model price of P^m=104.03 then

$$\Delta P = \frac{103.78}{104.03} - 1 = -0.24\% ,$$

is the quoted price displacement, i.e. the missing return in the computation of contributions using the model price. Note that for the typical evaluation period we have two quoted price displacements: one at the beginning and one at the end of the period.

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In the following subsections we are going to define the contribution adjustments incrementally in two successive approximations. We derive the following results for interest-rate only securities and for risky bonds at the same time.

4.1 First-order quoted contributions

Consider a security with a model price P^m for which we know the interest-rate duration LD, the spread duration SD, the credit spread s^m and model interest rate r^m. In section 3 we define the flattened pricing function $P^z(r,s)$, see equation (55), to summarize the price dependence from a single interest rate r and the credit spread s. We do not specify the time t because the results obtained are valid for both at the beginning of the period, with $t{=}t_0$, and the end, where $t{=}t_1$.

We assume that there is a *quoted pricing function* P^q such that when evaluated at the *quoted spread* s^q and at the *quoted interest rate* r^q, returns the quoted price P^q:

$$P^q = P^q(r^q, s^q)\,. \tag{64}$$

The actual values of s^q and r^q are unknown, however based on our assumptions they must satisfy the above equation.

We do not know the quoted pricing function $P^q(r,s)$, however we assume that it can be approximated by a Taylor series expansion of the flattened pricing function $P^z(r,s)$ in a neighborhood of the model risk drivers:

$$P^q(r,s) = P^m \cdot \left[1 - SD\,(s - s^m) - LD\,(r - r^m)\right]\,. \tag{65}$$

Substituting equation (64) into this expression and using definition (63) for the price displacement ΔP, we obtain

$$\Delta P = -SD\,(s^q - s^m) - LD\,(r^q - r^m)\;, \tag{66}$$

i.e. an equation for the two variables s^q and r^q. Since we only have one equation and two unknowns we need to make some further assumptions in order to determine the values of both s^q and r^q.

For interest-rate only securities we assume that the difference between the quoted price and the model price can only be caused by an interest rate r^q that is different from the model interest rate r^m, i.e. we have

$$r^q = r^m + \Delta r^q \quad \text{and} \quad s^q = s^m + 0\,,$$

for some real number Δr^q. From equation (66) we obtain an equation for Δr^q,

$$\Delta P = -LD \cdot \Delta r^q\,, \tag{67}$$

that we solve, assuming $LD \neq 0$, to obtain

$$\Delta r^q = -\frac{\Delta P}{LD}\,,$$

Note that we can think of equation (67) as defining the single (first order) contribution to the price displacement as being strictly an interest-rate contribution.

For risky bonds as discussed earlier, we assume that the difference between the quoted price and the model one is due to a credit spread s^q different from s^m. Therefore in this case we set

$$r^q = r^m + 0 \quad \text{and} \quad s^q = s^m + \Delta s^q\,,$$

so that, using a derivation similar to that of interest-rate only securities, we obtain

$$\Delta s^q = -\frac{\Delta P}{SD}\,.$$

In the generic case we assume the quoted price to be due to a mixture of variations from the interest rate and the credit spread:

$$r^q = r^m + \Delta r^q \qquad \text{and} \qquad s^q = s^m + \Delta s^q\,. \tag{68}$$

Again note that the quoted risk drivers are defined as a perturbation with respect to the model ones. Since we have only one equation to determine both Δr^q and Δs^q we assume them to be proportional to the same number δ with *risk-driver performance weights* w_s and w_r so that

$$\Delta s^q = w_s \cdot \delta \qquad \text{and} \qquad \Delta r^q = w_r \cdot \delta\,, \tag{69}$$

with $w_r = 1 - w_s$. In this case equation (66) becomes

$$\Delta P = -SD \cdot \Delta s^q - LD \cdot \Delta r^q\,, \tag{70}$$

so that we have

$$\Delta P = -\delta \cdot (SD \cdot w_s + LD \cdot w_r) \ ,$$

and we can solve for δ:

$$\delta = -\frac{\Delta P}{SD \cdot w_s + LD \cdot w_r} \ . \tag{71}$$

This equation summarizes the case for interest-rate securities, for which we have w_s=0 and w_r=1 and the case of bonds with w_s=1 and w_r=0.

Once we computed the value of δ from equation (71) we can look at expression (70) under a different light: there are two (first order) contributions to the quoted displacement ΔP. The first contribution, namely $-SD \cdot \Delta s^q$, is a spread contribution, while the second one, i.e. $-LD \cdot \Delta r^q$, is an interest-rate contribution. We can then add these two term to the model contributions to obtain a decomposition for the quoted return.

4.2 Second-order quoted contributions

The contribution model described in section 1 is accurate to the second order in the risk drivers, therefore it seems natural to extend the computation of the contribution adjustments to the same order.

In this formulation we assume the second-order quoted pricing function to be given by

$$\begin{aligned} P^q(r,s) \ = \ & P^m \cdot [1 - SD\,(s - s^m) - LD\,(r - r^m) \\ & + \frac{CV^s}{2}\,(s - s^m)^2 + \frac{CV^r}{2}\,(r - r^m)^2 \\ & + CV^{rs} \cdot (s - s^m) \cdot (r - r^m)] \ , \end{aligned} \tag{72}$$

where all the model-price sensitivities have been defined in section 3.

Using the above definition we approximate the quoted pricing function with the second-order Taylor expansion of the flattened function $P^z(r,s)$ around the model risk drivers s^m and r^m. Since $P^q(r,s)$ provides the (approximate) quoted price when evaluated for s=s^q and r=r^q, we substitute definitions (68) into equation (64) to obtain

$$\begin{aligned} \Delta P \ = \ & -SD \cdot \Delta s^q - LD \cdot \Delta r^q \\ & + \frac{CV^s}{2}\,(\Delta s^q)^2 + \frac{CV^r}{2}\,(\Delta r^q)^2 \\ & + CV^{rs} \cdot (\Delta s^q) \cdot (\Delta r^q) \ . \end{aligned} \tag{73}$$

Then with the help of definition (69) we obtain

$$\begin{aligned} \Delta P \ = \ & -SD \cdot w_s \cdot \delta - LD \cdot w_r \cdot \delta \\ & + \frac{CV^s}{2}\,(w_s \cdot \delta)^2 + \frac{CV^r}{2}\,(w_r \cdot \delta)^2 \\ & + CV^{rs} \cdot (w_s \cdot \delta) \cdot (w_r \cdot \delta) \ , \end{aligned} \tag{74}$$

that we can write as

$$\Delta P = A \cdot \delta^2 + B \cdot \delta \, , \tag{75}$$

with

$$A \ = \ \frac{CV^s}{2} w_s^2 + \frac{CV^r}{2} w_r^2 + CV^{rs} \cdot w_s \cdot w_r \ ,$$

and

$$B = -SD \cdot w_s - LD \cdot w_r \ .$$

Expression (75) is a quadratic equation for δ and in order to solve it we need to consider few separate cases.

The first case is when $A \simeq 0$ and $B \simeq 0$, in which case there is no solution.

The second case is when $A \simeq 0$ and $B \neq 0$ so that the quadratic equation is degenerate and effectively we are back to the same equation as subsection 4.1 and δ is given by equation (71).

In the third case we have $A \neq 0$ so that equation (75) is not degenerate, there are two sub-cases: the first one is for $B \simeq 0$. In this sub-case we have no way to distinguish between the two solutions and we simply settle for their average:

$$\delta = 0 \, .$$

In the second sub-case, ie. when $B \neq 0$, we can only find the solutions when the discriminant of the quadratic equation is positive, i.e. when

$$B^2 + 4 \cdot A \cdot \Delta P > 0 \, . \tag{76}$$

Here there are two solutions, δ_+ and δ_-, given by

$$\delta_\pm = \frac{-B \pm \sqrt{B^2 + 4 \cdot A \cdot \Delta P}}{2 \cdot A} \, . \tag{77}$$

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In order to decide which solution we can accept we look at the slope of the function $A \cdot \delta^2 + B \cdot \delta$, i.e.

$$B + 2 \cdot A \cdot \delta\,, \tag{78}$$

on each of the two solutions δ_+ and δ_- and we accept the solution that has the same sign as the slope for δ=0 (which is actually the sign of B). Since both solutions are obtained from equation (77) we have

$$B + 2 \cdot A \cdot \delta_\pm = \pm\sqrt{B^2 + 4 \cdot A \cdot \Delta P}\,, \tag{79}$$

hence we choose δ_+ when B>0 and δ_- when B<0. The motivation behind this choice is quite straightforward: we prefer a solution that has a slope with the same sign as that computed on the model price (oriented according to the weights w_r and w_s).

Rejecting high price displacements In some rare cases, for example when ΔP is rather large, even the second-order approximation does not provide reliable adjustments results. In these cases unfortunately the current model does not give any insight into the computation of the return contributions and one should simply content himself with the model-price contributions.

A natural limit for ΔP is given by inequality (76), that we can re-write as

$$-4 \cdot A \cdot \Delta P < B^2\,.$$

However, since A can be either positive or negative, this inequality only limits ΔP on one side. To obtain and inequality that limits ΔP from both side we can take the absolute value of the expression above:

$$|\Delta P| < \frac{\left|B^2 - 4 \cdot A \cdot C\right|}{4 \cdot |A|}\,.$$

Using a bit of algebra it can be shown that this expression is equivalent to the following inequality:

$$|\delta| < \frac{|B|}{2 \cdot |A|}\,.$$

In this way we prescribe a maximum absolute displacement allowed for the relative difference between the model price and the quoted price.

Second order quoted contributions After we find the value of δ we can give a different meaning to equation (73) as describing the return contributions in going from P^m to P^q. By matching each terms in equation (73) with the corresponding terms of equation (12), according to the corresponding sensitivities, we write

$$\Delta P = C^s_{\text{1st}} + C^r_{\text{1st}} + C^s_{\text{2nd}} + C^r_{\text{2nd}} + C^{rs}_{\text{2nd}}\,, \tag{80}$$

with

$$C^s_{\text{1st}} = -SD \cdot \Delta s^q \qquad \text{and} \qquad C^r_{\text{1st}} = -LD \cdot \Delta r^q\,,$$

for the first-order contributions, and

$$\begin{aligned} C^s_{\text{2nd}} &= \frac{CV^s}{2}\,(\Delta s^q)^2\,, \\ C^r_{\text{2nd}} &= \frac{CV^r}{2}\,(\Delta r^q)^2\,, \\ C^{rs}_{\text{2nd}} &= CV^{rs} \cdot \Delta r^q \cdot \Delta s^q\,, \end{aligned}$$

for the second-order contributions. In this way we can clearly identify and give meaning to the contributions of the quoted price that are not already accounted among the model-return contributions

Note that when w_s is either zero or one, so that w_r is either one or zero, the term C^{rs}_{2nd} is always zero.

4.3 Quoted-price return contributions

In this subsection we gather together all the pieces from the quoted displacements (there is one at the beginning of the period and one at the end), the risk-driver model return, and the carry terms to obtain the quoted-price return contributions. We write the quoted-price return Ret^q as

$$\begin{aligned} \text{Ret}^q &= \frac{P^q_1 + C_1 - P^q_0}{P^q_0} \\ &= \frac{1}{P^q_0}\Big[P^q_1 - P^z(r^m_1, s^m_1, t_1) \\ &\qquad + P^z(r^m_1, s^m_1, t_1) - P^z(r^m_0, s^m_0, t_1) \\ &\qquad + P^z(r^m_0, s^m_0, t_1) + C_1 - P^z(r^m_0, s^m_0, t_0) \\ &\qquad + P^z(r^m_0, s^m_0, t_0) - P^q_0\Big]\,, \end{aligned}$$

and identify each term on the right-hand side of this equation. The first term:

$$\frac{1}{P^q_0}\left[P^q_1 - P^z(r^m_1, s^m_1, t_1)\right] = \frac{P^m_1}{P^q_0} \cdot \Delta P_1\,,$$

is proportional to ΔP_1, see equation (63), the return from the model price to the quoted price at $t{=}t_1$, and can be written as a sum of contributions as described by equation (80). The second term

$$\frac{1}{P_0^q}\left[P^z(r_1^m, s_1^m, t_1) - P^z(r_0^m, s_0^m, t_1)\right] = \frac{P_0^m}{P_0^q}\cdot C^{\text{driv}}\,,$$

is proportional to the risk-driver contribution C^{driv} defined in equation (8) and its contributions were defined in equation (12). The third term

$$\frac{1}{P_0^q}\Big[P^z(r_0^m, s_0^m, t_1) + C_1 - P^z(r_0^m, s_0^m, t_0)\Big] = \frac{P_0^m}{P_0^q}\cdot C^{\text{carry}}\,,$$

is proportional to the carry contribution C^{carry} defined in equation (7) that can be split into a convergence and a coupon contribution as described at the end of subsection 1.4. The fourth and last term

$$\frac{1}{P_0^q}\left[P^z(r_0^m, s_0^m, t_0) - P_0^q\right] = -\frac{P_0^m}{P_0^q}\cdot \Delta P_0\,,$$

is proportional to ΔP_0, the return from the model price to the quoted price at $t{=}t_0$, and its contributions can be computed using equation (80).

Explicit computation of contributions The split into additive contributions of each of the four terms above allows us to write the quoted return as

$$\frac{P_1^q - P_0^q}{P_0^q} = \dot{T}_{\text{cnv}}\cdot\Delta t + \dot{T}_{\text{cpn}}\cdot\Delta t + \dot{C}^s_{1^{\text{st}}} + \dot{C}^r_{1^{\text{st}}} + \dot{C}^s_{2^{\text{nd}}} + \dot{C}^r_{2^{\text{nd}}} + \dot{C}^{rs}_{2^{\text{nd}}} + \varepsilon\,, \tag{81}$$

where the adjusted contributions are computed as follows: the adjusted convergence sensitivity $\dot{T}_{\text{cnv}}$

$$\dot{T}_{\text{cnv}} = \frac{P_0^m}{P_0^q}\,T_{\text{cnv}} = \frac{\hat{P}_0^m}{P_0^q}\,\hat{T}_D\,, \tag{82}$$

the adjusted coupon sensitivity $\dot{T}_{\text{cpn}}$

$$\dot{T}_{\text{cpn}} = \frac{P_0^m}{P_0^q}\,T_{\text{cpn}} = \frac{\hat{P}_0^m}{P_0^q}\,\hat{C}_D\,, \tag{83}$$

the adjusted first-order spread contribution

$$\dot{C}^s_{1^{\text{st}}} = -\frac{P_1^m}{P_0^q}SD_1\cdot\Delta s_1^q + \frac{P_0^m}{P_0^q}SD_0\cdot\Delta s_0^q - \frac{P_1^m}{P_0^q}SD_1\cdot\Delta s^m\,,$$

the adjusted first-order interest-rate contribution

$$\dot{C}^r_{1^{\text{st}}} = -\frac{P_1^m}{P_0^q}LD_1\cdot\Delta r_1^q + \frac{P_0^m}{P_0^q}LD_0\cdot\Delta r_0^q - \frac{P_1^m}{P_0^q}LD_1\cdot\Delta r^m\,,$$

the adjusted second-order spread contribution

$$\dot{C}^s_{2^{\text{nd}}} = \frac{P_1^m}{P_0^q}\frac{CV_1^s}{2}\left(\Delta s_1^q\right)^2 - \frac{P_0^m}{P_0^q}\frac{CV_0^s}{2}\left(\Delta s_0^q\right)^2 - \frac{P_1^m}{P_0^q}\frac{CV_1^s}{2}\cdot(\Delta s^m)^2\,,$$

the adjusted second-order interest-rate contribution

$$\dot{C}^r_{2^{\text{nd}}} = \frac{P_1^m}{P_0^q}\frac{CV_1^r}{2}\left(\Delta r_1^q\right)^2 - \frac{P_0^m}{P_0^q}\frac{CV_0^r}{2}\left(\Delta r_0^q\right)^2 - \frac{P_1^m}{P_0^q}\frac{CV_1^r}{2}\cdot(\Delta r^m)^2\,,$$

and finally the adjusted second-order interest/spread contribution

$$\dot{C}^{rs}_{2^{\text{nd}}} = \frac{P_1^m}{P_0^q}CV_1^{rs}\cdot\Delta r_1^q\cdot\Delta s_1^q - \frac{P_0^m}{P_0^q}CV_0^{rs}\cdot\Delta r_0^q\cdot\Delta s_0^q - \frac{P_1^m}{P_0^q}CV_1^{rs}\cdot\Delta r^m\cdot\Delta s^m\,.$$

In the above expressions we use the subscript i to denote that the computations are evaluated for $t{=}t_i$, with $i{=}0{,}1$. Also recall that

$$\Delta r^m = r_1^m - r_0^m = r_1^z - r_0^z + \delta_1^z - \delta_0^z$$

and

$$\Delta s^m = s_1^m - s_0^m\,.$$

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4.4 Adjusted sensitivities

We can simplify the above expressions for the fixed-income contributions by defining the following *adjusted* sensitivities:

$$\dot{LD}_i = \frac{P_i^m}{P_0^q}\, LD_i \quad \text{and} \quad \dot{SD}_i = \frac{P_i^m}{P_0^q}\, SD_i\,,$$

for the first-order and

$$\dot{CV}_i^s = \frac{P_i^m}{P_0^q}\, CV_i^s \quad \text{and} \quad \dot{CV}_i^r = \frac{P_i^m}{P_0^q}\, CV_i^r\,,$$

together with

$$\dot{CV}_i^{rs} = \frac{P_i^m}{P_0^q}\, CV_i^{rs}\,,$$

for the second-order, where i=0,1.

Using the results of subsection 1.5, we can write the adjusted sensitivities in terms of the clean-price sensitivities, for i=0,1:

$$\dot{LD}_i = \frac{\hat{P}_i^m}{P_0^q}\, \hat{LD}_i\,, \qquad \dot{SD}_i = \frac{\hat{P}_i^m}{P_0^q}\, \hat{SD}_i\,,$$

$$\dot{CV}_i^s = \frac{\hat{P}_i^m}{P_0^q}\, \hat{CV}_i^s\,, \qquad \dot{CV}_i^r = \frac{\hat{P}_i^m}{P_0^q}\, \hat{CV}_i^r\,,$$

and

$$\dot{CV}_i^{rs} = \frac{\hat{P}_i^m}{P_0^q}\, \hat{CV}_i^{rs}\,.$$

Note that in the above expressions the denominator of the adjustment factor is always P_0^q while the numerator $\hat{P}_i^m$, i.e. the clean model price, depends on the date t_0 or t_1.

We then substitute these definitions into the previous expressions for the contributions, see equation (81), to obtain

$$\dot{C}_{1\text{st}}^s = -\dot{SD}_1 \left(\Delta s_1^q + \Delta s^m\right) + \dot{SD}_0 \cdot \Delta s_0^q\,,$$

$$\dot{C}_{1\text{st}}^r = -\dot{LD}_1 \cdot \left(\Delta r_1^q + \Delta r^m\right) + \dot{LD}_0 \cdot \Delta r_0^q\,,$$

$$\dot{C}_{2\text{nd}}^s = \frac{\dot{CV}_1^s}{2} \left[\left(\Delta s_1^q\right)^2 - \left(\Delta s^m\right)^2\right] - \frac{\dot{CV}_0^s}{2} \left(\Delta s_0^q\right)^2\,,$$

$$\dot{C}_{2\text{nd}}^r = \frac{\dot{CV}_1^r}{2} \left[\left(\Delta r_1^q\right)^2 - \left(\Delta r^m\right)^2\right] - \frac{\dot{CV}_0^r}{2} \left(\Delta r_0^q\right)^2\,,$$

$$\begin{aligned}\dot{C}_{2\text{nd}}^{rs} &= \dot{CV}_1^{rs} \left[\Delta r_1^q \cdot \Delta s_1^q - \Delta r^m \cdot \Delta s^m\right] \\ &\quad - \dot{CV}_0^{rs} \cdot \Delta r_0^q \cdot \Delta s_0^q\,,\end{aligned}$$

i.e. the somewhat simplified expressions for the fixed-income contributions.

4.5 Contributions of custom interest-rate curves

The interest rate r^z obtained from the curve used to compute the model price is sometimes different from the custom rate r^c that the user prefers as a reference rate. In order to provide contributions that are in line with the user expectations we need to move parts of the interest-rate contributions to the spread contributions.

Hence consider a custom interest rate changing from r_0^c at the beginning of the period to r_1^c at the end. The model interest-rate increment Δr^m,

$$\begin{aligned}\Delta r^m &= r_1^z - r_0^z + \delta_1^z - \delta_0^z \\ &= r_1^c - r_0^c + \delta_1^z - \delta_0^z \\ &\qquad + r_1^z - r_0^z - r_1^c + r_0^c \\ &= \Delta r^c + \Delta s^c\,,\end{aligned}$$

can be written as the sum of the custom interest-rate variation Δr^c, defined as

$$\Delta r^c = r_1^c - r_0^c + \delta_1^z - \delta_0^z\,,$$

and a custom added spread Δs^c:

$$\Delta s^c = r_1^z - r_1^c + r_0^c - r_0^z\,.$$

Notice that the Δr^c also contains the zero-shift increase $\delta_1^z - \delta_0^z$ that is needed to compute returns for non-flat interest-rate curves.

We substitute Δr^m=Δr^c+Δs^c into equation (81) and re-arrange the terms so that Δr^c is gathered among the interest-rate terms and Δs^c ends up in the spread terms. We thus obtain

$$\begin{aligned}\text{Ret}^q &= \dot{T}_{\text{cnv}} \cdot \Delta t + \dot{T}_{\text{cpn}} \cdot \Delta t + \\ &\quad \ddot{C}_{1\text{st}}^s + \ddot{C}_{1\text{st}}^r + \ddot{C}_{2\text{nd}}^s + \ddot{C}_{2\text{nd}}^r + \ddot{C}_{2\text{nd}}^{rs} + \varepsilon\,,\end{aligned} \tag{84}$$

with the same carry sensitivities $\dot{T}_{\text{cnv}}$ and $\dot{T}_{\text{cpn}}$ defined earlier. We then define the custom first-order spread contribution as

$$\ddot{C}^s_{1\text{st}} = -\dot{SD}_1 \left(\Delta s^q_1 + \Delta s^c\right) + \dot{SD}_0 \cdot \Delta s^q_0 - \dot{LD}_1 \cdot \Delta s^c \,,$$

the custom first-order interest-rate contribution as

$$\ddot{C}^r_{1\text{st}} = -\dot{LD}_1 \left(\Delta r^q_1 + \Delta r^c\right) + \dot{LD}_0 \cdot \Delta r^q_0 \,,$$

the custom second-order spread contribution as

$$\ddot{C}^s_{2\text{nd}} = \frac{\dot{CV}^s_1}{2}\left[\left(\Delta s^q_1\right)^2 - \left(\Delta s^m\right)^2\right] - \frac{\dot{CV}^s_0}{2}\left(\Delta s^q_0\right)^2 - \frac{\dot{CV}^r_1}{2}\cdot\left(\Delta s^c\right)^2 - \dot{CV}^{rs}_1 \cdot \Delta s^c \cdot \Delta s^m \,,$$

the custom second-order interest-rate contribution as

$$\ddot{C}^r_{2\text{nd}} = \frac{\dot{CV}^r_1}{2}\left[\left(\Delta r^q_1\right)^2 - \left(\Delta r^c\right)^2\right] - \frac{\dot{CV}^r_0}{2}\left(\Delta r^q_0\right)^2 \,,$$

and finally the custom second-order interest/spread contribution as

$$\ddot{C}^{rs}_{2\text{nd}} = \dot{CV}^{rs}_1\left(\Delta r^q_1 \cdot \Delta s^q_1 - \Delta r^c \cdot \Delta s^m\right) - \dot{CV}^{rs}_0 \cdot \Delta r^q_0 \cdot \Delta s^q_0 - \dot{CV}^r_1 \cdot \Delta r^c \cdot \Delta s^c \,.$$

Equation (84) can be used to determine the return contributions according to the user custom choice of the interest rate. Note that in principle we could further split the Δr^c term into a purely flat contribution $r^c_1 - r^c_0$ and a *curvy* contribution $\delta^z_1 - \delta^z_0$, however we leave this model extension to future works.

5 Summary of the contribution computations

In this subsection we write a summary of the steps needed to compute the fixed-income contributions of the quoted price return as described by equation (84).

Our goal is to compute the fixed-income contributions for a security in the period between t_0 and t_1. We use the notation to add a 0 subscript (i.e. we write something like "$\square_0$") for the quantities computed at the beginning of the period, i.e. for $t{=}t_0$, and a subscript 1 (writing something like "$\square_1$") for those computed at the end, i.e. for $t{=}t_1$. We describe in each of the following subsections a step in the recipe for computing the performance contributions and write the final result in subsection 5.9.

5.1 The model pricing function

We assume that for the given security we have available a quantitative model, a numerical method, and the market data to compute the bond model price P^m both at $t{=}t_0$ and at $t{=}t_1$. For interest-rate only securities, as described in subsection 3.1 we assume the model price to be given by equation (48):

$$P^m_i = P(L_i(T), \mathcal{M}, t_i) \,.$$

Similarly for bonds where the credit spread s^m is a relevant risk driver we compute the model price as

$$P^m_i = P\left(L_i(T), s^m_i, \mathcal{M}, t_i\right) \,.$$

Recall that, in these equations, $\mathcal{M}$ generically represents the market data, t_i is the evaluation date, and $L_i(T)$ the interest-rate term structure observed at time $t{=}t_i$.

5.2 The flattened pricing function

In principle the pricing function depends on the whole interest-rate term structure $L_i(T)$, so that, for each cash flow occurring at date T we compute the discount factor using a different interest rate. In practice however we want to summarize the effects of the whole curve into a single effective interest-rate rate r^m_i. In order to use the two-factor model we replace the interest-rate curve $L_i(T)$ with a flat term structure $Z_i(r, T)$ depending on a single variable r. At the beginning of subsection 2.1 we show how to build a flat zero-rate term structures for different compounding conventions.

For interest-rate securities, see subsection 3.1, we define the *flattened pricing function* P^z as

$$P^z(r, s, t_i) = P(Z_i(r, T), \mathcal{M}, t_i) \,,$$

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while for risky bonds, as shown in subsection 3.2, we define it as

$$P^z(r,s,t_i) = P(Z_i(r,T),s,\mathcal{M},t_i)\,.$$

Since the flattened pricing function $P^z(r,s,t_i)$ depends at most on two market variables r and s, together with the current time t_i, we can use the results of subsection 1.4 to compute the fixed-income contributions of returns.

5.3 The model risk-driver displacements

The flattened pricing function depends on two risk drivers r and s, and needs to be evaluated at the beginning of the period to provide P_0^m and at the end to give P_1^m.

From the data used by the quantitative model we need to extract the credit spreads s_0^m and s_1^m used to compute the model bond prices at t_0 and t_1. We then compute the model spread displacement as

$$\Delta s^m = s_1^m - s_0^m\,.$$

We assume that for the interest-rate securities the credit spread is always s_i^m=0 for both i=0,1.

The interest rate r^m should then be chosen so that the flattened pricing function provides exactly the model price, i.e. we have

$$P^z(r^m,s^m,t_i) = P_i^m\,, \tag{85}$$

for i=0,1. These equations can be solve to obtain the model interest-rates r_0^m and r_1^m, so that we compute the model interest-rate displacement as

$$\Delta r^m = r_1^m - r_0^m\,.$$

5.4 The zero-curve increment

Given the interest-rate curves $L_0(T)$ and $L_1(T)$ we can compute, as shown at the beginning of section 3, the zero-rates r_0^z and r_1^z at the bond maturity, so that the discount factor computed using r_i^z on this flat interest rate is the same as that computed using the full interest-rate curve. We then define the zero-rate displacement as

$$\Delta r^z = r_1^z - r_0^z\,.$$

Given the zero interest rates r_i^z's, with i=0,1, we define the zero shifts at time i as in equation (51), i.e.

$$\delta_0^z = r_0^m - r_0^z \qquad \text{and} \qquad \delta_1^z = r_1^m - r_1^z\,.$$

5.5 The custom-curve increment

When the user wants to evaluate the interest-rate contributions on interest rates r_i^c's that are different from r_i^z's we need to compute the custom interest-rate variation as

$$\Delta r^c = r_1^c - r_0^c + \delta_1^z - \delta_0^z\,,$$

and add a custom spread

$$\Delta s^c = r_1^z - r_1^c + r_0^c - r_0^z\,.$$

Both term Δr^c and Δs^c are necessary for the full computation of contributions when the user uses a custom interest-rate term structure.

5.6 The time sensitivities

As shown in subsection 4.3 the adjusted convergence sensitivity $\dot{T}_{\text{cnv}}$ and the adjusted coupon sensitivity $\dot{T}_{\text{cpn}}$ are defined by equations (82) and (83). Recall that the clean-price carry sensitivity $\hat{T}_D$ is defined in equation (27) and that the coupon sensitivity $\dot{T}_{\text{cnv}}$ is given by expression (28) so that

$$\dot{T}_{\text{cnv}} = \frac{\hat{P}_0^m}{P_0^q}\,\hat{T}_D \qquad \text{and} \qquad \dot{T}_{\text{cpn}} = \frac{\hat{P}_0^m}{P_0^q}\,\hat{C}_D\,,$$

where

$$\hat{T}_D = \frac{1}{\hat{P}_0^m}\,\frac{\hat{P}^z(r_0,s_0,t_1) - \hat{P}^z(r_0,s_0,t_0)}{\Delta t}\,,$$

and

$$\hat{C}_D = \frac{1}{\hat{P}_0^m}\,\frac{C_1 + A(t_1) - A(t_0)}{\Delta t}\,.$$

Note that the convergence term can also be computed using the full pricing function, instead of the flattened one, because

$$\hat{P}(Z_0(r,T),s_0,\mathcal{M},t_0) = \hat{P}^z(r_0,s_0,t_0)$$

by definition of r_0 and s_0 and

$$\hat{P}(Z_0(r,T), s_0, \mathcal{M}, t_1) \simeq \hat{P}^z(r_0, s_0, t_1)$$

with a very small error (otherwise we would have to keep track of an extra contribution coming from the difference of these two values).

5.7 The risk-driver sensitivities

Given the flattened pricing function we compute the model sensitivities from the derivatives of $P^z(s,r,t_i)$ evaluated for $r{=}r_i^m$ and $s{=}s_i^m$, with i=0,1. The interest-rate and the spread durations are defined as

$$LD_i = -\frac{1}{P_i^m}\frac{\partial P_i^z}{\partial r} \qquad \text{and} \qquad SD_i = -\frac{1}{P_i^m}\frac{\partial P_i^z}{\partial s}\,.$$

Similarly we define the interest-rate convexity as

$$CV_i^r = \frac{1}{P_i^m}\frac{\partial^2 P_i^z}{\partial r^2}\,,$$

together with the spread convexity CV_i^s and the rate/spread convexity CV_i^{rs} defined as

$$CV_i^s = \frac{1}{P_i^m}\frac{\partial^2 P_i^z}{\partial s^2} \qquad \text{and} \qquad CV_i^{rs} = \frac{1}{P_i^m}\frac{\partial^2 P_i^z}{\partial r \partial s}\,.$$

Note that the derivatives of the flattened pricing function can be either computed analytically, for example for simple bonds as described in section 2, or numerically for complex bonds, as described in section 3.

5.8 The risk-driver quoted adjustments

In order to account for the quoted price displacements ΔP_i, defined in equation (63), we assume the quoted pricing functions P_i^q to be equal to the quadratic approximation of P_i^z around the model risk drivers. We then define the quoted interest rate as

$$r^q = r^m + \Delta r^q = r^m + w_r \cdot \delta$$

and the quoted credit spread as

$$s^q = r^m + \Delta r^q = r^m + w_s \cdot \delta\,.$$

The weights w_s and w_r in the previous equations will be kept generic with the only constraint that

$$w_s = 1 - w_r \qquad \text{and} \qquad 0 \le w_r \le 1\,.$$

It is typical to set w_s=1 and w_r=0 for risky bonds, while w_s=0 and w_r=1 for interest-rate only securities.

As described in subsection 4.2, see specifically definition (72), we define the quoted pricing function P^q as a second-order approximation of the flat pricing function. By matching the actual quoted prices P_i^q with that coming from $P^q(r_i^q, s_i^q, t_i)$ we obtain a second-degree equation for the displacement δ_i, as shown in equation (75). We then proceed, as described in subsection 4.2, to compute the two solutions δ_0 and δ_1, possibly rejecting high price displacements as already specified earlier, so that we can compute the risk-driver quoted adjustments

$$\Delta r_i^q = w_r \cdot \delta_i \qquad \text{and} \qquad \Delta s_i^q = w_s \cdot \delta_i$$

to the interest rate and the credit spreads both at $t{=}t_0$ and at $t{=}t_1$. With this explicit expression for Δr_i^q and Δs_i^q we are able to use the decomposition expression (80) to split the relative return of the quoted price with respect to the model price as the sum of fixed-income contributions.

5.9 The performance contributions

As shown in subsections 4.3 and 4.4 we can simplify the expressions for the fixed-income contributions by defining the adjusted sensitivities in terms of the clean-price sensitivities. Then, as described in subsection 4.5, gathering all the elements computed in the previous subsections, we write the single-period two-factor return contributions as

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$$\begin{aligned}
\text{Ret}^q &= && (86)\\
\{\text{carry}\} \quad & \dot{T}_{\text{cnv}} \cdot \Delta t + \dot{T}_{\text{cpn}} \cdot \Delta t \\
\{\text{spread}\} \quad & -\dot{SD}_1 \cdot \Delta s_1^q + \dot{SD}_0 \cdot \Delta s_0^q - \dot{SD}_1 \cdot \Delta s^m - \dot{LD}_1 \cdot \Delta s^c \\
\{\text{rate}\} \quad & -\dot{LD}_1 \left(\Delta r_1^q + \Delta r^c\right) + \dot{LD}_0 \cdot \Delta r_0^q \\
\{\text{spread}\} \quad & +\frac{\dot{CV}_1^s}{2}\left[\left(\Delta s_1^q\right)^2 - \left(\Delta s^m\right)^2\right] - \frac{\dot{CV}_0^s}{2}\left(\Delta s_0^q\right)^2 - \frac{\dot{CV}_1^r}{2} \cdot \left(\Delta s^c\right)^2 - \dot{CV}_1^{rs} \cdot \Delta s^c \cdot \Delta s^m \\
\{\text{rate}\} \quad & +\frac{\dot{CV}_1^r}{2}\left[\left(\Delta r_1^q\right)^2 - \left(\Delta r^c\right)^2\right] - \frac{\dot{CV}_0^r}{2}\left(\Delta r_0^q\right)^2 \\
\{\text{spread/rate}\} \quad & +\dot{CV}_1^{rs}\left(\Delta r_1^q \cdot \Delta s_1^q - \Delta r^c \cdot \Delta s^m\right) - \dot{CV}_0^{rs} \cdot \Delta r_0^q \cdot \Delta s_0^q - \dot{CV}_1^r \cdot \Delta r^c \cdot \Delta s^c \\
\{\text{residual}\} \quad & +\varepsilon \,,
\end{aligned}$$

where in the first line on the right-hand-side of this equation we have the carry terms, on the second line we have the first-order spread contributions, on the third line we notice the first-order interest-rate contributions, on the fourth line we write the second-order spread contributions, on the fifth line the second-order interest-rate contributions, on the sixth line the cross spread/rate contributions, and finally on the last line the residual.

5.10 Simplified fixed-income contributions

According to equation (86) there are a couple of dozen contributions to the quoted return of a generic security. While some of these terms might be very small for some securities, in general they are all needed to minimize the residual. In certain cases the users might trade precision for simplicity as they prefer a smaller number of contributions. One way to reduce the number of terms is to assume that the sensitivities are almost equal at the beginning and at the end of the period: i.e.

$$\dot{LD}_1 \simeq \dot{LD}_0 \qquad \text{and} \qquad \dot{SD}_1 \simeq \dot{SD}_0 \,,$$

and similarly for the convexities. We can further reduce the number of contributions by assuming that all adjusted convexities are approximately equal to their average

$$\overline{CV}_i = \frac{\dot{CV}_i^r + \dot{CV}_i^s + \dot{CV}_i^{rs}}{3} \,.$$

In other words we assume that

$$CV_i^r \simeq CV_i^s \simeq CV_i^{rs} \simeq \overline{CV}_i \,.$$

Using the above approximations we can write the quoted return as

$$\begin{aligned}
\text{Ret}^q &= && (87)\\
\{\text{carry}\} \quad & \dot{T}_{\text{cnv}} \cdot \Delta t + \dot{T}_{\text{cpn}} \cdot \Delta t \\
\{\text{spread}\} \quad & -\dot{SD}_1\left(\Delta s_1^q - \Delta s_0^q + \Delta s^m\right) - \dot{LD}_1 \cdot \Delta s^c \\
\{\text{rate}\} \quad & -\dot{LD}_1\left(\Delta r_1^q - \Delta r_0^q + \Delta r^c\right) \\
\{\text{convexity}\} \quad & +\frac{\overline{CV}_1}{2}\left[\left(\Delta y^m\right)^2 + \left(\Delta y_1^q\right)^2 - \left(\Delta y_0^q\right)^2\right] \\
\{\text{residual}\} \quad & +\varepsilon \,,
\end{aligned}$$

where the model-yield increment is defined as

$$\Delta y^m = \Delta r^c + \Delta s^c + \Delta s^m$$

and the yield-adjustment increases are written as

$$\Delta y_i^q = \Delta s_i^q + \Delta r_i^q$$

for i=0,1.

fixed-rate coupon bond In the case of a fixed-rate coupon bond equation (87) can be further simplified because, as note earlier, we have identical first-order sensitivities:

$$\dot{LD}_1 = \dot{SD}_1 = \dot{MD}\,.$$

In this case we write equation (87) as

$$\begin{aligned}\mathrm{Ret}^q \;=\; & \dot{T}_{\mathrm{cnv}} \cdot \Delta t + \dot{T}_{\mathrm{cpn}} \cdot \Delta t \\ & -\dot{MD}_1 \left(\Delta y_1^q - \Delta y_0^q + \Delta y^m\right) \\ & +\frac{\overline{CV}_1}{2} \left[(\Delta y^m)^2 + \left(\Delta y_1^q\right)^2 - \left(\Delta y_0^q\right)^2\right] \\ & +\varepsilon\,,\end{aligned} \tag{88}$$

i.e. the sum of the two carry terms, the three first order contributions, the three convexity contributions, and the residual.

6 Conclusions

In the last few subsections we show a summary of the steps needed to compute the performance contributions of the single-period single-security quoted return both in the full form of equation (86) and in the simplified form of expression (87). In both equations the contributions are additive and each of them has a specific financial meaning.

Earlier in the paper we choose to aggregate the carry terms, the interest-rate contributions, and the credit-spread components. That was just one choice since we can also choose gather the contributions in other ways. For example we can aggregate all terms that arise from the quoted adjustment at t_0, those from the adjustment at t_1, those coming from the model contributions, and finally we would be left with the carry terms and the residual. As an other example we can place the term $-\dot{LD}_1 \cdot \Delta s^c$ either with the credit terms, when the custom interest-rate term structure is used, or with the interest-rate terms when the model curve is used.

In presence of contributions for multiple periods and for more than one security in a portfolio, we should also aggregate the single-period single-security quoted return contributions over longer periods and at sector or portfolio level. However we leave the details of these aggregations to the references in the bibliography.

As a final remark we recall that the carry contribution C^{carry}, defined in equation (7), is computed exactly, i.e. without the need to resort to a Taylor expansion. As it turns out, as seen in reference [4], we can also split the risk-driver contribution C^{driv}, see definition (8), exactly into three terms: one for the interest rates, one for the credit spread, and a cross term depending on variations of both risk drivers. Furthermore, as described in the same reference, the same technique can be generalized to compute the exact contributions for securities with a pricing function depending on any number of risk drivers.

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