Risk historical simulations of VIX futures spreads

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Abstract

In this work we describe a method to perform risk simulations of VIX futures, according to the historical-simulation model. We assume a stochastic volatility mean-reverting constant elasticity of variance process to model the VIX dynamics. Following non-arbitrage argument, the market expectation of VIX futures price results in a function of three financial variables: the spot VIX, the long-run mean of the mean-reverting VIX process, and a time scale parameter. For each trading day in the period 2011-2012 we collect the closing VIX futures market quotes across all available maturities and calibrate the three financial variables using an ad-hoc least-squares procedure. In this way we can extract the VIX futures term-structure for each day in the sample period. We then compute historical scenarios for all financial variables and we apply these scenarios to a calendar spread position on VIX futures to simulate its value. Finally, from the distribution of the simulated spread values we derive the P&L (profit and loss) strip which is used to compute risk figures.

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1 Introduction

In 1993 the Chicago Board Options Exchange (CBOE) introduced the Volatility Index, in short VIX, which was originally designed to measure the market’s expectation of 30-day volatility implied by at-the-money S&P 100 Index (OEX) option prices. Ten years later in 2003, CBOE together with Goldman Sachs, updated the VIX definition to reflect a new way to measure expected volatility, one that continues to be widely used by financial theorists, risk managers, and volatility traders alike. The new VIX index is thus based on the S&P 500 Index (SPX for short) and estimates the 30-day expected volatility by averaging the weighted prices of SPX puts and calls over a wide range of strike prices. The goal of the VIX index is to capture the volatility of the SPX over the next month implicit in stock index option prices. Formally, it is the square root of the risk-neutral variance of the SPX over the next 30 calendar days, reported on an annualised basis. Despite this rather technical definition, both financial market participants and the media pay a lot of attention to its movements. To some extent its popularity is due to the fact that VIX changes are negatively correlated to changes in stock prices. Indeed, the volatility goes up in periods of market decline. Moreover, investors trade options on the S&P 500 to buy protection in periods of market turmoil, which increases the value of the VIX. For that reason, some commentators refer to it as the market’s fear gauge, even though a high VIX value does not necessarily imply negative future returns.

The new VIX was also intended to be used as an underlying index for VIX derivatives. Indeed, in 2004 the CBOE introduced a specific exchange, namely the CBOE Futures Exchange (CFE), precisely to provide exchange-traded volatility derivatives and the first VIX futures contract was introduced. In 2006, following the success of VIX futures, CBOE launched also VIX options. The negative correlation of volatility to stock market returns is well documented and suggests a diversification benefit to including volatility in an investment portfolio. VIX futures and options are designed to deliver pure volatility exposure and as a consequence diversification benefits from volatility are exploitable in the easiest way. VIX options and VIX futures are among the most actively traded contracts at CBOE and CFE, averaging close to 140,000 contracts combined per day. The high interest in these products is mainly due to their ability to hedge the risks of investments in the S&P 500 index.

Although these assets certainly offer additional investment and hedging opportunities, their correct use requires reliable valuation models that adequately
capture the features of the underlying volatility index. Many models have been proposed in literature in the attempt to correctly model the VIX index and to derive reliable pricing functions for VIX derivatives. The most important feature to retain is undoubtedly the mean reversion of volatility and as a consequence the mean-reverting stochastic volatility class of models seems to be the best fit for this purpose. Even though some authors in literature suggest that a jump process should be included in VIX modeling, evidence is not conclusive and reasonable results can still be obtained without jumps. As far as VIX derivatives pricing is concerned the evidence is twofold: the inclusion of a jump process seems to be not necessary for VIX futures pricing, whereas it seems to improve the evaluation model for VIX options.

In this work we take advantage of the pricing model discussed in Dupoyet, Daigler, and Chen (2011), i.e. we assume the spot VIX process to be mean-reverting constant elasticity of variance (CEV). As we focus on VIX futures pricing, we decide to discard the jump component, according to the conclusions of the Dupoyet, Daigler, and Chen paper. In this theoretical framework we describe a method to perform risk simulations of VIX futures. In particular we assume a calendar spread position on VIX futures and we follow the historical approach to perform risk simulations. A calendar spread involves a long position on futures with a longer maturity and a short position on futures with a nearer maturity. The VIX futures price results in a function of three risk factors, i.e. the financial variables that are source of risk. For each day in the period 2011-2012 we extract the VIX futures term-structure, calibrating the three financial variables. We then compute historical scenarios for all financial variables and we apply these scenarios to simulate the value of the calendar spread position on VIX futures. Finally, from the distribution of the simulated spread values we derive the P&L (profit and loss) strip which is used to compute risk figures.

The remainder of this work is organised as follows. In Section 2 we discuss the volatility issues, focusing on the concept of implied volatility and its smile. Section 3 provides a detailed description of the VIX index and some related topics. In Section 4, after a review of the literature on the VIX modeling, we handle the mean-reverting CEV process and the resulting VIX futures price. Section 5 explains step-by-step the calibration procedure of the VIX futures pricing function. In Section 6 we describe how to carry out the risk historical simulations of VIX futures spreads, providing the results. Finally, Section 7 summarizes and concludes.
2 A primer on volatility

2.1 Back to basics

Volatility has always played a key role in finance since the second half of the twentieth century, when, finance itself became a distinct field of economics. The first studies about portfolio selection (Markowitz 1952, Sharpe 1964) laid the basis for all the following developments and set the core of finance: the trade-off between risk and expected return. Various measures have been proposed to figure out the risk component of this trade-off, but the standard deviation of an asset’s return, better known as volatility, has undoubtedly been the most commonly used measure of risk. Apart from being an important input into portfolio theory, asset volatilities are of even greater significance for derivatives pricing. Now financial institutions are more and more engaged in volatility trading and volatility has become an asset class itself, as several asset managers argue to be successful in performance improvement exploiting diversification benefits from volatility.

Even though most would agree upon the importance of volatility, its real meaning is not so clear among many investors and finance students. So some preliminaries could be useful to grasp the issue. We use the term volatility to refer to the standard deviation, usually denoted with $\sigma$, (or variance, $\sigma^2$), and we compute this standard deviation or variance from a sample of observations as

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (R_t - \bar{R})^2. \quad (1)$$

where $\bar{R}$ is the mean return, and obviously $\sigma$ is the square root of (1).

The sample standard deviation statistic $\sigma$ is a distribution free parameter representing the second moment characteristic of the sample, i.e. the dispersion measure of the sample. It can be related to a regular distribution, such as the Gaussian, in which case the probability density can be derived analytically, or calculated from any irregular shape distribution, in which case the probability density have to be derived empirically. In the continuos time setting, $\sigma$ is a scale parameter that multiplies or reduces the size of the fluctuations generated by the standard Wiener process. Depending on the dynamic of the underlying stochastic process and whether or not the parameters are time varying very different shapes of returns distributions may result. So a meaningful use of $\sigma$ as a risk measure requires it to refer to a distribution or a pricing dynamic. As
far as the link between volatility and risk is concerned, this is not so strong. In fact risk is more often associated with negative returns, whereas standard deviation and most measures of dispersion do not make distinction. The Sharpe ratio, for example, defined as return in excess of risk-free rate divided by standard deviation, is frequently used as an investment performance measure and incorrectly penalizes occasional high returns. Markowitz himself suggested the idea of semi-variance, which only uses the squared of returns below the mean. This and other alternatives to volatility or variance have not been widely used because they are not easy to apply in portfolio construction.

2.2 Stylized facts about volatility

When most of students and investors discuss about volatility as risk measure they have in mind a normal distribution for returns. First of all because this is the main assumption we find in every finance textbook “for simplicity sake”. Moreover the three cornerstones of finance, the model of portfolio construction by Markowitz, the Capital Asset Pricing Model by Sharpe and the Black Sholes model for pricing European equity vanilla options rest upon the strong assumption of normal / log-normal distribution, besides the hypothesis of efficient and frictionless markets.

Financial markets have experienced some events that question this assumption, especially after the 1987 crash. Several salient features about financial time series and volatility are now well documented as stylized facts: fat tail distributions of risky asset returns, volatility clustering, that is “large moves follow large moves and small moves follow small moves”, asymmetry, asset prices reactions to “bad news” are more pronounced than reactions to “good news”, mean reversion, volatility is pulled-back to its long run mean when it is above or below this level, and comovements of volatilities across assets and financial markets. More recent studies find correlation among volatility (auto-correlation) is stronger than that among returns and both tend to increase during bear markets and financial crises. High values of auto-correlations coefficient imply that a shock in the volatility process will have a long-lasting impact.

All things considered, volatility matters a lot for pricing, asset and risk management. Investors and portfolio managers have certain levels of risk which they can bear. A good forecast of the volatility of asset prices over the investment holding period is the starting point for assessing investment risk. Volatility is
a very important variable in the pricing of derivative securities. To price an option we need to know the volatility of the underlying asset from now until the option expires. In fact the market convention is to list option prices in terms of volatility units. Moreover, volatility has an even more significant role in determining the payoff of volatility derivatives, such as variance swaps and VIX derivatives. So it is a necessary work to make good volatility forecasting. A good forecasting is all but easy work because we need models that can account for the aforementioned stylized facts.

2.3 A brief review of volatility forecasting

Practitioners and researchers have been engaged in the challenging task of volatility forecasting and a plethora of models and techniques have been issued. Basically one can collect them in two categories: time series models and market estimations. In a nutshell we make distinction according to two kind of approaches. Time series volatility models use the historical information to formulate volatility forecasts, whereas the second approach derives market estimates of future volatility from traded option prices.

All time series volatility models, in their different specifications, capture volatility persistence or clustering, and some of them take into account volatility asymmetry, too. The main hurdle is not to construct a theory based model for financial returns that is heteroskedastic, i.e. accounting for non uniform variability of returns. Forecasting volatility is to some extent “more art than science”. It is reasonable to expect that models with some forecasting ability will be those catching the main features of volatility found with actual returns. Now an explanation of forecasting models is provided with some clarifications.

The first way to make predictions about volatility is to gather information from past standard deviations. The simplest historical price model is the random walk model, where \( \sigma_{t-1} \) is used as a forecast for \( \sigma_t \). Following the same idea, we have the Historical Average method, the simple Moving Average method, the Exponential Smoothing method and the Exponentially Weighted Moving Average method. The Historical Average method makes use of all historical standard deviations while the Moving Average method discards the older ones. Similarly, the Exponential Smoothing method uses all historical estimates, and the Exponentially Weighted Moving Average method uses only the more recent ones. But unlike the previous two, the two exponential methods give greater
weights to the more recent volatility estimates. The Simple Regression method expresses volatility as a function of its past values and an error term, therefore this method is principally autoregressive (AR). If past volatility errors are also included, we have the ARMA (Autoregressive Moving average) model for volatility. Introducing a differencing order $I(d)$, we get ARIMA (AR integrated MA) when $d = 1$ and ARFIMA (AR fractionally integrated MA) when $d < 1$. Finally, we have the Threshold Autoregressive model, where the thresholds separate volatility into states with independent simple regression models and noise processes for volatility in each state.

A more sophisticated class of time series models is that of ARCH (Autoregressive Conditional Heteroskedasticity) type models (extensively surveyed in e.g. Bollerslev, Engle and Nelson (1994)). These models do not make use of sample standard deviations, but formulate conditional variance, $h_t$, of returns via maximum likelihood procedure. Moreover, because of the way ARCH models are constructed, $h_t$ is known at time $t - 1$. So the one-step ahead forecast is readily available. Forecasts that are more than one step ahead can be formulated based on an iterative procedure. The first example among these models is ARCH($q$) (Engle 1982) where $h_t$ is a function of $q$ past squared returns. In GARCH (Generalized ARCH)($p, q$) (Bollerslev 1986 and Taylor 1986) past values of $h_t$ on $p$ lags are included. Empirical analysis finds GARCH models to work better than ARCH, and GARCH(1, 1) is the most popular structure for many financial time series. The EGARCH (Exponential GARCH) model (Nelson 1991) specifies conditional variance in logarithmic form, in order to obtain only positive values for variance without imposing estimation constraint. With suitable conditioning of the parameters, this specification captures the stylized fact that a negative shock leads to a higher conditional variance in the subsequent period than a positive shock would. Other models that allow for non-symmetrical dependencies are the TGARCH (Threshold GARCH), which is similar to the GJR-GARCH (Glosten, Jagannathan and Runkle 1993), the QGARCH (Quadratic GARCH) and various other non-linear GARCH versions (for a review see Franses and van Dijk (2000)). Another way to model asymmetry in volatility is to think about two volatility regimes, high and low, in order to take advantage of a regime switching framework. In this sense RS-GARCH (Regime switching GARCH) (Hamilton 1989) models produce different values for volatility persistence depending on whether it is in high or low volatility regime. A GARCH model features an exponential decay in the autocorrelation of conditional variances. However, it has been noted that squared and abso-
lute returns of financial assets typically have serial correlations that are slow to decay, like in an $I(d)$ process. A shock in the volatility series seems to have very “long memory” and impact on future volatility over a long horizon. The Integrated GARCH (IGARCH) model (Engle and Bollerslev 1986) captures this effect but a shock in this model impacts upon future volatility over an infinite horizon, and the unconditional variance does not exist for this model. This gives rise to FIGARCH$(p,d,q)$ and FIEGARCH$(p,d,q)$ (Baillie, Bollerslev and Mikkelsen (1996)).

### 2.4 Stochastic volatility

In a completely different framework we find Stochastic volatility (SV) models, i.e. volatility is let to move in a random fashion. Volatility is subject to a source of innovations that may or may not be related to those that drive returns. Modeling volatility as a stochastic variable immediately leads to fat tail distributions for returns. Autoregressive term in the volatility process introduces persistence, and correlation between the two innovative terms in the volatility process and the return process produces volatility asymmetry. The volatility noise term makes the SV model a lot more flexible, but as a result the SV model has no closed form, and hence cannot be estimated directly by maximum likelihood. However one can cope with this problem through analytical solutions, numerical integration or Monte Carlo integration. This kind of models offer a noticeable benefit, directly linked to the second volatility forecasting approach that is the options-based one. The Black Sholes model, as already said, rests upon some counterfactual assumptions, among those we find the assumption of constant volatility. SV models relaxing this assumption can account for the “volatility smile”, a well-known puzzle produced by the Black Sholes model, providing in this sense an absolutely not trivial benefit.
2.5 Volatility smile

The Black-Sholes model for pricing European equity options assumes stock price has the following dynamics

\[ dS = \mu S dt + \sigma S dz. \]  

which is a geometric Brownian motion. From Ito’s lemma, the logarithmic of stock price has the following dynamics

\[ d\ln S = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dz. \]

which means that stock price has a lognormal distribution or the logarithmic of stock price has a normal distribution. Using a risk-less hedge argument, Black-Sholes proved that under certain assumptions, option prices could be derived using a risk neutral valuation relationship where all derivative assets generate only risk-free returns. The assumptions include constant volatility \( \sigma \), short sell with full use of proceeds, no transaction costs or taxes, perfect divisibility of securities, no dividend before option maturity, no arbitrage, continuous trading and a constant risk-free interest rate \( r \). Apart from the constant volatility assumption, the violation of any of the remaining assumptions will result in the option price being traded within a band instead of at the theoretical price.

Option traders and investors do not believe the assumptions made by this model. Markets are not complete and frictionless and the empirical distribution of the underlying price is, typically, too heavy-tailed for the price to be generated by a geometric Brownian motion. The Black-Sholes option pricing formula states that option price at time \( t \) is a function of the price of the underlying asset \( S_t \), the strike price \( K \), the risk-free interest rate \( r \), the time to option maturity \( T \) and \( \sigma \), the volatility of the underlying asset over the period from \( t \) to \( T \),

\[ C(S,t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2). \]  

\[ P(S,t) = -S_t N(-d_1) + Ke^{-r(T-t)} N(-d_2). \]

where \( C(S,t) \) is the call option price, \( P(S,t) \) is the put option price,

\[ d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}. \]
$$d_2 = d_1 - \sigma \sqrt{T - t}.$$  

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. Given that $S_t$, $K$, $r$, and $T$ are observable, once the market has produced a price (either a quote or a transaction price) for the option, we could extract the value of $\sigma$ that the market used as an input. Such a volatility estimate is called option implied volatility. Since the reference period is from $t$ to $T$ in the future, option implied volatility is often interpreted as a market’s expectation of volatility over the option’s maturity, i.e. the period from $t$ to $T$. Given that each asset can have only one $\sigma$, it is a well-known puzzle that options of the same time to maturity but differing in strikes appeared to produce different implied volatility estimates for the same underlying asset. Volatility smile, smirk and sneer are names given to nonlinear shapes of implied volatility plots against strike price. Many theoretical explanations have been proposed, but distributional assumption and stochastic volatility seem to better solve this puzzle. \(^1\) To understand how Black-Scholes distributional assumption produces volatility smile, we need to make use of the positive relationship between volatility and option price, and the put-call parity

$$C_t + Ke^{r(T-t)} = P_t + S_t.$$  \hspace{1cm} (7)

which is a general non-arbitrage relationship and is independent from any distributional assumption, establishing the positive relationship between call and put option prices. Since implied volatility is positively related to option price, the put-call parity suggests there is also a positive relationship between implied volatilities derived from call and put options that have the same strike price and the same time to maturity. As mentioned before Black-Scholes requires stock price to be log-normally distributed or the logarithmic stock returns to have a normal distribution. There is now widely documented empirical evidence that risky financial asset returns have leptokurtic tails. In the case where strike price is very high, the call option is deep-out-of-the-money and the probability for this option to be exercised is very low. Nevertheless, a leptokurtic right tail will give this option a higher probability, than that from a normal distribution, for the

\(^1\)Other explanations are based on market microstructure, measurement errors, and investor risk preference.
terminal asset price to exceed the strike price and the call option to finish in
the money. This higher probability leads to a higher call price and a higher
Black-Scholes implied volatility at high strike. Next, we look at the case when
strike price is low. First note that the option value has two main components:
intrinsic value and time value. Intrinsic value reflects how deep the option is
in the money. Time value reflects the amount of uncertainty before the option
expires; hence it is most influenced by volatility. Deep-in-the-money call option
has high intrinsic value and little time value, and a small amount of bid-ask
spread or transaction tick size is sufficient to perturb the implied volatility esti-
mation. Out-of-the-money put option has a close to nil intrinsic value and the
put option price is due mainly to time value. Again because of the thicker tail
on the left, we expect the probability that out-of-the-money put option finishes
in the money to be higher than that for a normal distribution. Hence the put
option price (and hence the call option price through put-call parity) should
be greater than that predicted by Black-Scholes. If we use Black-Scholes to in-
vert volatility estimates from these option prices, the Black-Scholes implied will
be higher than actual volatility. This results in volatility smile where implied
volatility is much higher at very low and very high strikes.

The thick tail and non-symmetrical distribution referred to in the previous
section could be a result of volatility being stochastic. First, we rewrite (2) as

\[ dS_t = \mu_s S_t dt + \sigma_t S_t dz_s. \]  

and now \( \sigma_t \) has its own dynamics

\[ d\sigma_t^2 = (\mu_v - \beta \sigma_t^2) dt + \sigma_v \sigma_t^2 dz_t. \]  

where \( \beta \) is the speed of the volatility process mean reverting to the long-run
average \( (\mu_v/\beta) \) and \( \sigma_v \) is the volatility of volatility, which, alone, is enough to
produce kurtosis and Black-Scholes volatility smile. In this framework we can
also model the correlation between the price process and the volatility process,
via the correlation parameter \( \rho \). When \( \rho < 0 \), large negative return corresponds
to high volatility stretching the left tail further into the left. On the other
hand, when return is very high, volatility is low, squashing the right tail nearer
to the centre. This will give rise to low implied volatility at high strikes and
volatility skew. The reverse is true when \( \rho > 0 \). Consensus emerges on the
degree of Black-Scholes pricing bias as a result of stochastic volatility. In the
case where volatility is stochastic and $\rho = 0$, Black-Scholes overprices near-the-money (NTM) or at-the-money (ATM) options and the degree of overpricing increases with maturity. On the other hand, Black-Scholes underprices both in- and out-of-the-money options. In terms of implied volatility, ATM implied volatility would be lower than actual volatility while implied volatility of far-from-the-money options (i.e. either very high or very low strikes) will be higher than actual volatility. The pattern of pricing bias will be much harder to predict if $\rho$ is not zero, there is a premium for bearing volatility risk, and if either or both values vary through time. Some of the early work on option implied volatility focused on finding an optimal weighting scheme to aggregate implied volatility of options across strikes (see Bates 1996 for a comprehensive survey of these weighting schemes). Since the graph of implied volatility against strikes can take many shapes, it is not likely that a single weighting scheme will remove all pricing errors consistently. For this reason ATM option implied volatility is often used for volatility forecast but not implied volatilities at other strikes. ATM option implied volatility is popular as forecast measure also because for most option contracts, ATM option has the largest trading volume.
3 The CBOE Volatility Index (VIX)

3.1 The “investor fear gauge”

Since its inception in 1993, the CBOE Volatility Index (VIX) has become the main benchmark for stock market volatility. It was revised in 2003 to accomplish with the introduction of exchange-traded VIX derivatives. The combined trading activity in VIX options and futures has grown to more than 140,000 contracts per day. Apart from this remarkable trading activity, attention has been devoted to VIX in recent years of financial crises. Financial news services have routinely reported the VIX (and still do) with the tag “investor fear gauge”. This investor fear is often misunderstood and as a consequence VIX suffers from misconception. VIX measures 30-day expected volatility of the S&P 500 Index. The computation is made intraday according to a weighting scheme for S&P 500 options implied volatilities across different strikes and maturities. From the discussion in subsection 2.5, we may deduce that the construction of VIX is an example of good practice, in the sense that CBOE Volatility Index accounts for a volatility skew. However there is a common aptitude to intend the high VIX levels, experienced during market crashes, as the fear of returns realizing lower than expected, instead of the fear of volatility higher than expected. Hedgers and investors usually buy index puts, as they bought insurance policies, when they are afraid of a potential drop in the stock market and the more they demand for portfolio insurance the higher the price. VIX is an indicator that reflects the price of portfolio insurance. The fact that the VIX spikes during periods of market turmoil well explains why it has become known as the “investor fear gauge”. There are two forces at play. Firstly, if expected market volatility increases (decreases), investors demand higher (lower) rates of return on stocks, so stock prices fall (rise). But VIX rises at a higher absolute rate when the stock market falls than when it rises, because of the increasing demand for index puts. So the second force at play is this portfolio insurance effect. In attempting to understand VIX, it is important to emphasize that it is forward-looking, measuring volatility that the investors expect to see. It is not backward-looking, measuring volatility that has been recently realized. VIX is implied by the current prices of S&P 500 index options and represents expected future market volatility over the next 30 calendar days. To shed light upon this issue we start with definition of the VIX index, also explaining the reasons behind the shift from the old VIX to the new VIX.
3.2 The old VIX (VXO)

To begin, VIX is an index, computed on a real-time basis throughout each trading day. VIX was introduced in 1993 with two purposes in mind. First, it was intended to provide a benchmark of expected short-term market volatility. To facilitate comparisons of the then-current VIX level with historical levels, minute-by-minute values were computed using index option prices dating back to the beginning of January 1986. This was particularly important since documenting the level of market anxiety during the worst stock market crash since the Great Depression, the October 1987 crash, would provide useful benchmark information in assessing the degree of market turbulence experienced subsequently. Second, VIX was intended to provide an index upon which futures and options contracts on volatility could be written. The original construction of VIX (see Whaley 1993) uses options data on S&P 100 index (OEX) to compute an average of the Black and Sholes option implied volatility with strike prices close to the current spot index level and maturities interpolated at about one month. With the switch to the new version, this old style index was renamed VXO, and CBOE still provides quotes on it.

Eight near-the-money options, four calls and four puts, at the two nearest maturities are chosen for the computation. When the time to the nearest maturity is within eight calendar days, the next two nearest maturities are used instead. At each maturity, the CBOE chooses two call and two put options at the two strike prices that straddle the spot level and are nearest to it. The CBOE first averages the two implied volatilities from the put and call at each strike price, and then linearly interpolates between the two average implied volatilities at the two strike prices to obtain the at-the-money spot implied volatility. The interpolated at-the-money implied volatilities at the two maturities are further interpolated along the maturity dimension to create a 22-trading day volatility, which constitutes the VXO. The Black-Scholes implied volatility is the annualized volatility that equates the Black-Scholes formula value to the options market quote. The annualization is based on an actual/365 day-counting convention. Instead of using this implied volatility directly, the CBOE introduced an artificial “trading-day conversion” in the calculation of VXO. Specifically, let $ATMV(t, T)$ denote the time-$t$ Black-Scholes at-the-money implied volatility as an annualized percentage with expiry date $T$. The CBOE converts this
percentage to “trading-day” volatility $TV(t, T)$ according to

$$TV(t, T) = \text{ATMV}(t, T)\sqrt{\frac{NC}{NT}}.$$  \hspace{1cm} (10)

where $NC$ and $NT$ are the number of actual calendar days and the number of trading days between time $t$ and the option expiry $T$ respectively. The CBOE converts the number of calendar days into the number of trading days according to the following formula

$$NT = NC - 2 \times \text{int}(NC/7).$$  \hspace{1cm} (11)

VXO represents an interpolated trading-day volatility at 22 trading days based on the two trading-day volatilities at the two nearest maturities ($TV(t, T_1)$ and $TV(t, T_2)$):

$$VXO_t = TV(t, T_1) \frac{NT_2 - 22}{NT_2 - NT_1} + TV(t, T_2) \frac{22 - NT_1}{NT_2 - NT_1}.$$  \hspace{1cm} (12)

where $NT_1$ and $NT_2$ denote the number of trading days between time $t$ and the two option expiry dates $T_1$ and $T_2$, respectively. Since each month has about 22 trading days, VXO represents a one-month at-the-money implied volatility estimate. Due to the trading-day conversion, VXO is about 1.2 times (i.e. $\sqrt{365/252}$) greater than historical volatility computed using trading day data. This makes it no longer comparable to annualized realized volatilities computed from index returns. Thus, the VXO computation methodology has drawn criticism from both academia and industry for its artificially induced upward bias.

### 3.3 The new VIX

In contrast to the old VXO, which is based on near-the-money Black-Scholes implied volatilities of OEX options, the CBOE calculates the new volatility index VIX using market prices instead of implied volatilities. It also uses S&P 500 (SPX) options instead of OEX options. The new VIX uses a new formula that is not adversely affected by the assumptions of the Black-Scholes option pricing model, especially the resultant implication of an equal volatility across all strike prices. CBOE has created a historical record for the new VIX, too, dating back to 1990. The components of VIX are near- and next-term put and call options, usually in the first and second SPX contract months. The general
model-free formula for the new VIX calculation at time \( t \) is

\[
VS(t, T) = \frac{2}{T-t} \sum_i \frac{\Delta K}{K_i^2} e^{r(T-t)} O_i(K_i, T) - \frac{1}{T-t} \left( \frac{F_i}{K_0} - 1 \right)^2. \tag{13}
\]

where \( T \) is the common expiry date for all of the options involved in this calculation, \( F_i \) is the time-\( t \) forward index level derived from co-terminal index option prices, \( K_i \) is the strike price of the \( i \)-th out-of-the-money option in the calculation, \( O_i(K_i, T) \) denotes the time-\( t \) mid-quote price of the out-of-the-money option at strike \( K_i \), \( K_0 \) is the first strike below the forward index level \( F_t \), \( r_t \) denotes the time-\( t \) risk-free rate with maturity \( T^3 \), and \( \Delta K_i \) denotes the interval between strike prices, defined as \( \Delta K_i = (K_{i+1} - K_i)/2 \). The formula uses only out-of-the-money options except at \( K_0 \), at which \( O_t(K_0, T) \) represents the average of the call and put option prices at this strike. The last term represents the adjustment needed to convert the \( K_0 \) in-the-money call into an out-of-the-money put using put-call parity. The calculation involves all available call options at strikes greater than \( F_t \) and all put options at strikes lower than \( F_t \). Therefore, essentially the entire tradable range of the implied volatility smile (smirk) is included in the VIX calculation. Since the calculation is based on bid and ask prices, the implied volatilities do not change as option transactions prices change from the bid to the ask price. Theoretically, this new calculation procedure for the VIX yields an index that could be hedged using a weighted combination of options. The bids of these options must be strictly positive to be included. At the extreme strikes of the available options, the definition for the interval \( \Delta K \) is modified as follows: \( \Delta K \) for the lowest strike is the difference between the lowest strike and the next higher strike. Likewise, \( \Delta K \) for the highest strike is the difference between the highest strike and the next lower strike. VIX is a combination of the information reflected in the prices of all of the selected options. The contribution of a single option to the VIX value is proportional to \( \Delta K \) and the price of that option, and inversely proportional to the square of

2 The VIX calculation measures time to expiration, \( T - t \), in calendar days and divides each day into minutes in order to replicate the precision that is commonly used by professional option and volatility traders.

3 The risk-free interest rate is the bond-equivalent yield of the U.S. T-bill maturing closest to the expiration dates of relevant SPX options. As such, the VIX calculation may use different risk-free interest rates for near- and next-term options.

4 As volatility rises and falls, the strike price range of options with non-zero bids tends to expand and contract. As a result, the number of options used in the VIX calculation may vary from month-to-month, day-to-day and possibly, even minute-to-minute.

5 Once two puts/calls with consecutive strike prices are found to have zero bid prices, no more puts/calls with lower/higher strikes are considered for inclusion.
the option’s strike price. To determine the forward index level \( F_t \), the CBOE chooses a pair of put and call options with prices that are the closest to each other. Then, the forward price is derived via the put-call parity relation:

\[
F_t = e^{-r(T-t)}(C_t(K, T) - P_t(K, T)) + K. \tag{14}
\]

\( VS(t, T) \) is calculated at two of the nearest maturities of the available options, \( T_1 \) and \( T_2 \). Then, the CBOE interpolates between \( VS(t, T_1) \) and \( VS(t, T_2) \) to obtain a \( VS(t, T) \) estimate at 30-days to maturity. The VIX represents an annualized volatility percentage of this 30-day \( VS \), using an actual/365 day-counting convention:

\[
VIX_t = 100 \sqrt{\frac{365}{30} \left[ \frac{(T_1 - t)VS(t, T_1) \left( \frac{NC_2 - 30}{NC_2 - NC_1} \right) + (T_2 - t)VS(t, T_2) \left( \frac{30 - NC_1}{NC_2 - NC_1} \right)}{NC_2 - NC_1} \right]}. \tag{15}
\]

where \( NC_1 \) and \( NC_2 \) denote the number of actual days to expiration for the two maturities. When the nearest time to maturity is less than one week, the CBOE switches to the next nearest maturity in order to avoid microstructure effects, i.e. to minimize pricing anomalies that might occur close to expiration. The annualization follows the actual/365 day-counting convention and does not suffer from the artificial upward bias incurred in the VIX calculation.

### 3.4 Volatility and variance swap rate approximations

The VXO is essentially an average estimate of the one-month at-the-money Black-Scholes implied volatility, with an artificial upward bias induced by the trading-day conversion. Academics and practitioners often regard at-the-money implied volatility as an approximate forecast for realized volatility. However, since the Black-Scholes model assumes constant volatility, there is no direct economic motivation for regarding the at-the-money implied volatility as the realized volatility forecast beyond the Black-Scholes model context. We can show, under general market settings, an economic interpretation for at-the-money implied volatility in a theoretical framework which goes beyond the Black-Scholes model. The time-\( t \) at-the-money implied volatility with expiry at time \( T \) represents an accurate approximation of the conditional risk-neutral expectation of
the return volatility during the time period $[t, T]$:

$$ATMV(t, T) \approx E^Q_t[RVol_{t,T}],$$

where $E^Q_t[\cdot]$ denotes the expectation operator under the risk-neutral measure $Q$ and $RVol_{t,T}$ denotes the realized return volatility in annualized percentages over the time horizon $[t, T]$. This result gives new economic meanings to VXO. Indeed, if we re-adjust the upward bias induced by the trading-day conversion, the VXO approximates the volatility swap rate with a one-month maturity. Volatility swap contracts are traded actively over the counter on major currencies and some equity indexes. At maturity, the long side of the volatility swap contract receives the realized return volatility and pays a fixed volatility rate, which is the volatility swap rate. A notional dollar amount is applied to the volatility difference to convert the payoff from volatility percentage points to dollar amounts. Since the contract costs zero to enter, the fixed volatility swap rate equals the risk-neutral expected value of the realized volatility. It is worth noting that although the at-the-money implied volatility is a good approximation of the volatility swap rate, the payoff on a volatility swap is notoriously difficult to replicate. Carr and Lee (2003) derive hedging strategies for volatility swap contracts that involve dynamic trading of both futures and options.

The new VIX squared approximates the conditional risk-neutral expectation of the annualized realized return variance over the next 30 days:

$$VIX^2_t \approx E^Q_t[RV_{t,t+30}],$$

with $RV_{t,t+30} = RVol^2_{t,t+30}$ denoting the annualized return variance between $[t, t+30]$. Hence, $VIX^2_t$ approximates the 30-day variance swap rate. Variance swap contracts are actively traded over the counter on major equity indexes. At maturity, the long side of the variance swap contract receives a realized variance and pays a fixed variance rate, which is the variance swap rate. The difference between the two rates is multiplied by a notional dollar amount to convert the payoff into dollar payments. At the time of entry, the contract has zero value. Hence, by no-arbitrage, the variance swap rate equals the risk-neutral expected value of the realized variance. Although volatility swap payoffs are difficult to replicate, variance swap payoffs can be readily replicated as shown in Carr and Wu (2004). The trading strategy combines a static position in a continuum of options with a dynamic position in futures. The risk-neutral expected value of
the gains from dynamic futures trading is zero. The square of the VIX is a discretized version of the initial cost of the static option position required in the replication. The theoretical relation holds under very general conditions. We can think of the VIX as the variance swap rate quoted in volatility percentage points. Therefore, the new VIX index squared has a very concrete economic interpretation. It can be regarded either as the price of a portfolio of options or as an approximation of the variance swap rate.

3.5 Practical and theoretical motivations for the switch

As a premise we assert that critical to the timeliness and usefulness of any implied volatility index is to have it based on prices from a deep and active index option market. The original VIX index was based on the prices of S&P 100 (OEX) because, at the time, OEX options were the most actively-traded index options in the U.S. In contrast, the SPX option market was much more less active. A second feature of the original VIX was that it was based on the prices of only eight at-the-money index calls and puts, because at-the-money options were by far the most actively traded. Out-of-the-money options were less actively traded and frequently had stale price quotes and relatively wide bid/ask spreads. Including such quotes in the real-time computation of the VIX would reduce its timeliness and accuracy. Over the years since its inception, the structure of index option trading in the U.S. changed in two fundamental ways. First, the SPX option market became the most active index option market in the U.S. There are many attempts to explain why the trading volume shifted from one market to the other, but none is conclusive. Contributing factors include the fact that the S&P 500 index is better known, futures contracts on the S&P 500 are actively traded, and S&P 500 option contracts are European-style (i.e. exercisable only at expiration), making them easier to value, whereas OEX option contracts are American-style (i.e. exercisable within time to expiration). Second, trading motives of market participants in index option markets changed. In the early 1990s, both index calls and index puts had equally important roles in investor trading strategies. Over the following years, the index option market became dominated by portfolio insurers, who routinely buy out-of-the-money and at-the-money index puts for insurance purposes. The CBOE revised the VIX definition to account for both of these fundamental changes in index option market structure. First, they began to use SPX rather than OEX option
prices. Second, they began to also include out-of-the-money options in the index computation since out-of-the-money put prices, in particular, contain important information regarding the demands for portfolio insurance and, hence, market volatility. Including additional option series also helps make the VIX less sensitive to any single option price and hence less susceptible to manipulation. It is worth noting that the change from OEX option prices to SPX option prices had little to do with the return/risk properties of the indexes themselves. For all intents and purposes, the S&P 100 and S&P 500 index portfolios are perfect substitutes (see Whaley (2008)).

3.6 VIX derivatives

The economic meaning of the new VIX is much more concrete. Apart from the VXO upward bias induced by artificial trading-day conversion, which is easy to re-adjust, the fact that volatility swap payoffs are very difficult to replicate weakens the economic meaning of the VXO. One can readily replicate the variance swap payoffs, instead, using a static position in a continuum of European options and a dynamic position in futures trading (Carr and Wu 2005). Therefore, despite the popularity of VXO as a general volatility reference index, no derivative products have been launched on the VXO index. This phenomenon is quite unique among indexes, since almost all popular indexes have derivative products launched on them. In contrast, just a few months after the CBOE switched to the new VIX definition, they started planning to launch futures and options contracts on the new VIX. VIX futures started trading in 2004 on the CBOE Futures Exchange. We remember that VIX was intended also to provide an index upon which futures and options contracts on volatility could be written. The launch of VIX futures enjoyed great success. Motivated by the success of VIX futures, in 2006, the CBOE introduced options on the cash VIX. After the SPX index options, these VIX options are the CBOE’s most liquid option contract. Their popularity stems from the well-known negative correlation between VIX and SPX. Indeed, calls on VIX are often compared with puts on SPX. VIX options and VIX futures are among the most actively traded contracts at CBOE and CFE, averaging close to 140,000 contracts combined per day. The high interest in these products is mainly due to their ability to hedge the risks of investments in the S&P 500 index.

Specifically, on March 26, 2004, trading in futures on the VIX began on the
CBOE Futures Exchange (CFE). They are standard futures contracts, designed to reflect investors’ consensus view of future (30-day) expected stock market volatility. The CFE may list for trading up to nine near-term serial months and five months on the February quarterly cycle for the VIX futures contract. The underlying value of the VIX futures contract is the VIX multiplied by 1000$, the contract multiplier. VIX futures contracts cash settle on the Wednesday that is thirty days prior to the third Friday of the calendar month immediately following the month in which the contract expires (final settlement date). If the third Friday of the month subsequent to expiration of the applicable VIX futures contract is a CBOE holiday, the final settlement date for the contract shall be thirty days prior to the CBOE business day immediately preceding that Friday. Trading terminates on the business day immediately preceding the final settlement date of the VIX futures contract for the relevant spot month. The final settlement value for VIX futures shall be a Special Opening Quotation (SOQ) of VIX calculated from the sequence of opening prices of the options used to calculate the index on the settlement date. Settlement of VIX futures contracts will result in the delivery of a cash settlement amount on the business day immediately following the final settlement date. The cash settlement amount on the final settlement date shall be the final mark to market amount against the final settlement value of the VIX futures multiplied by $1000.

On February 24, 2006, European-style options on the VIX index were launched on the CBOE. Like VIX futures, they are cash settled according to the difference between the value of the VIX at expiration and their strike price. VIX options expire on the Wednesday that is thirty days prior to the third Friday of the calendar month immediately following the expiring month. CBOE may list up to six contract months, provided that the time to expiration is no greater than 12 months. The contract multiplier is 100$. Last trading date is the Tuesday prior to the expiration date of each month. As they are European-style options, they may be exercised only on the expiration date.

CBOE has extended VIX derivatives set introducing binary-options on VIX in 2008 and mini-VIX futures in 2009. CBOE VIX binary options are contracts that have an "all-or-nothing" payout depending on the settlement price of the VIX relative to the strike price of the binary option. Binary call options pay either a fixed cash settlement amount, if the VIX settles at or above the strike

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The opening price for any series in which there is no trade shall be the average of that option’s bid price and ask price as determined at the opening of trading. The final settlement value will be rounded to the nearest $0.01.

price at expiration or nothing at all, if the VIX settles below the strike price at expiration. Binary put options pay either a fixed cash settlement amount, if the VIX settles below the strike price at expiration or nothing at all, if the VIX settles above the strike price at expiration. Mini-VIX futures allows for a smaller contract size (VIX times 100$) than VIX futures and CFE may list for trading up to three near-term serial months contracts.
4 Evaluating the price of VIX futures

4.1 The stochastic-volatility option-pricing setting

The literature on the specification of the stochastic process that governs the dynamics of volatility in continuous time emerged in a stochastic volatility option-pricing context. In this setup, the underlying asset price and the volatility of the underlying asset returns are modeled jointly. In the late eighties, stochastic volatility option pricing models were developed by assuming a volatility process with continuous paths; the underlying asset price was also assumed to follow a diffusion process. In the late nineties, new type stochastic volatility option pricing models were introduced based on a jump-diffusion process for the underlying asset price and a diffusion volatility process. Recently, there have appeared option-pricing models where both the underlying asset price and the instantaneous volatility follow jump-diffusion processes.

The complete specification of the autonomous implied volatility process is important for two reasons, at least. Firstly, knowledge of the process that governs the evolution of implied volatility over time is particularly useful for volatility trading and hedging purposes. This is because the implied volatility is a reparametrisation of the market option price, and is used as an input to calculate the sensitivities of the option price with respect to various risk factors (hedge ratios). Secondly, the specification of the implied volatility process is necessary to price and hedge derivatives written on implied volatility itself, i.e volatility derivatives. Although volatility derivatives certainly offer additional investment and hedging opportunities, their correct use requires reliable valuation models that adequately capture the features of the underlying volatility index. We focus on VIX derivatives, searching for a proper stochastic volatility process to model the spot VIX dynamic. On any given point in time, the VIX represents the implied volatility of a synthetic option that incorporates information from all strikes and has constant time to maturity (30 days). This allows us to adopt the analogous processes that researchers have used to model the evolution of volatility in a stochastic-volatility option-pricing framework.
4.2 Current VIX derivatives pricing models

One line of research in the volatility literature focuses on finding and/or incorporating factors that explain changes in volatility over time. For example one could explain changes in implied volatility with various economic variables. Another line of research focuses on the volatility process directly, without necessarily investigating which underlying factors might be relevant and/or without linking the volatility process to a market index. Whereas the latter approach at first can seem incomplete relative to including potential underlying volatility-driving factors, in literature it has been demonstrated that relatively basic models are able to deliver hedging capabilities comparable to significantly more complex models. Additionally, whereas identifying latent, underlying volatility-driving factors can have many useful purposes, it can also make hedging more challenging. In particular, statistically identified factors are not tradable because they do not correspond to any actual tradable asset; they can therefore make hedging a complex endeavor.

Several theoretical approaches to price VIX derivatives appeared in the academic literature long before they could be traded. Specifically, Whaley (1993) priced volatility futures assuming that volatility follows a Geometric Brownian Motion (GMB). As a result, his model does not allow for mean reversion in volatility. The two most prominent mean-reverting models proposed so far have been the square root (SQR) process considered by Grünbichler and Longstaff (1996), and the log-normal Ornstein-Uhlenbeck (log-OU) process analyzed by Detemple and Osakwe (2000). Grünbichler and Longstaff have written the first theoretical paper on the valuation of futures and options on volatility. They derive a closed form solution for the futures price assuming volatility follows the dynamics laid out in Heston (1993) and others. Naturally, their model does not deal with the existing futures contract and its specifications. Several authors have recently assessed the empirical performance of these volatility derivative pricing models. In particular, Zhang and Zhu (2006) study the empirical validity of the SQR model by first estimating its parameters from VIX historical data, and then assessing the pricing errors of VIX futures implied by those estimates. Carr and Wu (2006) develop lower and upper bounds for the VIX futures price based on the forward volatility and forward variance swap rates. However the arbitrage bounds that they develop are very wide, significantly larger than the typical VIX futures bid-ask spread. Dotsis, Psychoyios, and Skiadopoulos (2007) explore alternative popular diffusion and jump-diffusion...
models and their ability to capture the dynamics of implied volatility indices. They also examine the pricing performance of the corresponding volatility futures pricing models. However, although they allow for jumps, the models are limited to either a mean-reverting square root process or a geometric Brownian motion. Surprisingly, this model yields reasonably good results, but the time span of their sample is perhaps too short for the mean-reverting features of the VIX to play any crucial role. Lin (2007) develops a VIX futures price model that incorporates jumps in both index return and volatility processes, stochastic volatility of the S&P 500 and risk-premium variables.

The historical evolution of VIX is characterized by swings from low to high levels, with a temporal pattern that shows mean reversion over the long run but displays strongly persistent deviations from the mean during extended periods. Both the SQR and the log-OU processes show some glaring deficiencies in capturing the strong persistence of the VIX, which produces large and lasting deviations of this index from its long run mean. In contrast, the implicit assumption in those models is that volatility mean reverts at a simple, non-negative exponential rate. Such a limitation becomes particularly apparent during bearish stock markets, in which volatility typically experiences large increases and remains at high levels for long periods. Unfortunately, the sample periods considered in the existing studies cover mostly the relatively long and quiet bull market that ended in the summer of 2007. Mencía and Sentana (2009) study the empirical ability of existing mean-reverting models to price VIX derivatives over a long sample that also includes data from 2007-2009 financial crisis. To obtain a good pricing performance during bull and bear markets they introduce relevant extensions: a time varying central tendency in the mean, jumps and stochastic volatility. A central tendency parameter allows the average volatility level to be time-varying while stochastic volatility permits a changing dispersion for the volatility index. They can conclude that adding jumps does not introduce any remarkable improvements, as far as VIX derivatives pricing is concerned, while stochastic volatility turns out to be more important to value VIX options and central tendency is crucial to price VIX futures. Dupoyet, Daigler and Chen (2011) propose and test a simple yet flexible constant elasticity of variance (CEV) model with and without jumps to price volatility futures, and to compare the model against the often used SQR formulation.
4.3 Factors affecting the VIX and VIX futures

There are a number of factors affecting the VIX index and thus the VIX futures. It is well known that volatility is stochastic and mean-reverting over time. Jumps can exist in volatility, although the frequency and size of the jumps is not well known. Certainly the most important feature to retain is mean reversion. The typical term structure of the VIX futures exhibits that the futures flatten out for longer-term expirations, showing how the long-term effect of mean reversion of the VIX index reflects on futures prices. In normal market conditions the average VIX futures term structure is typically upward sloping, indicating that the average short-term volatility is relatively low compared with the long-term mean level and that the volatility is increasing to the long-term higher level. During the global financial crisis in October and November 2008 the market was very volatile, hence the short-term volatility, such as VIX was high and the term structure of VIX futures was downward sloping. The phenomenon of downward sloping VIX futures volatility is consistent with the mean-reverting feature of the volatility. Since the long-term volatility approaches a fixed level, long-tenor VIX futures would be less volatile than short-tenor ones.

Establishing if the time-series of VIX contains jumps, or assuming that the VIX is a continuous process, is a key issue when modeling the spot VIX index. Since jumps in volatility affect individual option prices, they would directly affect the value of the VIX. Empirically determining if a given series is continuous or if that series contains discrete jumps is a challenging endeavor. The distinction between a continuous series and a series with jumps is a subtle one, since observed data are essentially discrete and therefore are made up of a series of small jumps (even when collected at high frequencies). Dupoyet et al. (2011) conduct a jump test on intraday VIX data and conclude that the inclusion of a jump component in the VIX model is necessary. In particular, actual market behavior shows that market volatility easily jumps to higher levels during sudden market turmoil, whereas it typically declines more gradually when the market returns to a normal state (see Mencia & Sentana (2009)). Hence, in order to have a jump-diffusion model consistent with this stylized feature of the market, one-sided jumps should be used. However, Dupoyet et al. (2011) show that models with jumps under-perform relative to simpler models (without jumps) for out-of-sample data, in spite of better in-sample accuracy. Thus, these out-of-sample results show the somewhat surprising lack of importance or sensitivity of the model to the jump component when actual futures pricing
is employed. Evidence from literature is not conclusive in this sense. Indeed, while Dupoyet et al. (2011) obtain reasonable yet better results without jumps, Lin (2007) concludes that jump models provide the best results.

4.4 Model specification

After a review of the literature on VIX and VIX futures we decide to base our work on the CEV formulation by Dupoyet et al. (2011). Therefore we allow flexibility in the evolution of the spot VIX process by incorporating mean-reversion and constant elasticity of variance. We decided to not include jumps, mainly because evidence in literature is not conclusive. Moreover, Dupoyet et al. (2011) support the jumps are not necessary to correctly model VIX futures prices. We assume the spot VIX process $V_t$ to evolve following this stochastic differential equation:

$$dV_t = (\alpha - \beta V_t)dt + \sigma_v(V_t)^\gamma dW_t,$$

where $\alpha$, $\beta$, $\gamma$, and $\sigma_v$ are constants, and $W_t$ is the standard Brownian motion. This equation describes a class of processes and has the great advantage to be easily handled mathematically. This general model, apart from accounting for mean-reversion of the VIX, allows for flexibility in the diffusion component, via the exponent $\gamma$. The parameter $\gamma$ controls the relationship between the underlying VIX index and its volatility, and is the central feature of this class of models. CEV models have been developed as an attempt to capture the so-called leverage effect, first discussed by Black (1976): given a fixed debt level, a decline in the equity level (stock price) increases the leverage of the firm and hence the risk for the stock (volatility). The negative correlation between S&P 500 returns and VIX captures this leverage effect. When $\gamma < 1$ we see exactly the leverage effect, commonly observed in equity markets, where the volatility of a stock increases as its price falls. Conversely, in commodity markets, we often observe $\gamma > 1$, the so-called inverse leverage effect, whereby the volatility of the price of a commodity tends to increase as its price increases. The general formulation nests three special cases:
1. For $\gamma = 0$, we have the Vasicek-type model.

2. For $\gamma = 1/2$, the model collapses to the well-known CIR model, initially developed by Cox, Ingersoll and Ross (1985) for interest rates, used by Heston (1993) for the instantaneous variance of stock prices, and adapted by Grünbichler and Longstaff (1996) to model the VIX process. The typical CIR model constrains the diffusion component to a square root process.

3. For $\gamma = 1$, we have a mean-reverting geometric Brownian process.

The mean-reverting CEV model was first proposed by Chan, Karolyi, Longstaff and Anthony (1992) to study the dynamics of short-term interest rates. It is similar to the CIR model but offers added flexibility in the diffusion component.

4.5 Derivation of VIX futures price

Futures pricing models are arbitrage cost-of-carry models that relate the underlying asset price to the futures price via holding the asset. This pricing strategy is based on the possibility to store the underlying asset. VIX futures have different characteristics, and, above all, the VIX cannot be stored. Since the VIX index is a risk neutral volatility forecast, not a directly traded asset, there is no cost of carry relationship between the price of the futures and the VIX. There is no convenience yield either, as in the case of futures on commodities. Moreover, holding the “underlying asset” for the VIX futures, i.e. a portfolio of S&P 500 options for almost all priced out-of-the-money calls and puts, is infeasible due to cost and liquidity issues. As stated by Grünbichler and Longstaff (1996, p. 988-989), discussing the results obtained from the valuation of a generic contingent claim on volatility, $V$, “It is important to note that the basic nature of our results is unaffected by whether $V$ can be expressed as a non-linear function of other security prices as in the case where $V$ is the implied volatility of an option. This is because $V$ cannot be replicated by a self-financing portfolio of the option and the index, even though a non-self-financing portfolio can be constructed which replicates $V$ exactly. The intuition for why this is true is related to the fact that all securities must earn the riskless rate of return in the risk-neutral economy. Thus, all self-financing portfolios must earn the riskless rate. However, the ‘expected return’ on $V$, can be either positive or negative and generally will not equal the riskless rate. This means $V$ cannot be
the value of a self-financing portfolio of securities and that standard hedging arguments are not applicable”. In other words VIX futures cannot be replicated. Therefore, absent any other market information, VIX futures must be priced according to a risk-neutral evolution of the VIX. This situation is similar, but not identical, to term structure models.

We assume the standard non-arbitrage pricing of VIX futures and, following Grünbichler and Longstaff (1996), we can compute the quoted price of futures contract expiring at time $T$ on the underlying spot VIX price $V_t$, at time $t$, as

$$F(V_t, t, T) = E^Q[V_T].$$

where $E^Q$ denotes the expectation under the risk-neutral measure $Q$. It can be shown (see the appendix in Dupoyet et al. (2011), discarding the jump component) that equations (18) and (19) imply a VIX futures price of

$$F(V_t, t, T) = V_t e^{-\beta(T-t)} + \frac{\alpha}{\beta}(1 - e^{-\beta(T-t)}).$$

Note that the $\gamma$ parameter disappears as it is eliminated when the expectation is taken in the derivation of the formula. As far as the risk premium is concerned, in literature it has been showed that model prices are a monotonic function of the price of risk. For example Dotsis et al. (2007) set the volatility risk premium equal to zero in their calibration exercise and they confirm the robustness of their results to the choice of different levels of the volatility risk premium. Taking advantage of this, we set the volatility price of risk equal to zero. Finally, in our model the volatility risk premium would enter via the $\beta$ parameter, i.e. the speed of mean reversion parameter. Everything held constant, different risk premium levels would simply monotonically affect the predicted futures price of the model.
4.6 Properties of VIX futures prices

VIX futures prices are exponentially weighted averages of the time-$t$ value of VIX and the long-run mean $\alpha/\beta$ of the process. If we set $t = 0$, we can rewrite (20) in function of the spot price of VIX, $V_0$

$$F(T) = V_0 e^{-\beta T} + \frac{\alpha}{\beta} (1 - e^{-\beta T}).$$

It is easy to note that as $T \to 0$, the futures price converges to the spot VIX level, and as $T \to \infty$, the futures price converges to $\alpha/\beta$. Volatility futures prices, in general, have interesting properties. For example, as $V \to 0$, the volatility futures price does not converge to zero. Thus, volatility futures prices are bounded above zero. The intuition for this is related to the mean reversion of the volatility process. When $V$ reaches zero, $V$ immediately is pulled back to positive values. Thus, the expected value of $V$ at time $T$ is strictly greater than zero even when the current value of $V$ is zero. This feature of volatility futures prices contrasts with those of futures prices on traded assets. Another important property of volatility futures prices is that their hedging effectiveness is a function of their maturity. In the limit as $T \to \infty$, futures prices approach the long-run mean $\alpha/\beta$ of the volatility process and are unaffected by the current value of $V$. For this reason, longer-term futures contracts may not be effective instruments for hedging short-term volatility risk. The intuition for this is again related to the mean reversion, as any change in the current value of $V$ is expected to be partially reversed prior to the expiration of the contract. Hence, volatility futures prices move sluggishly in response to volatility shocks, i.e. the mean-reversion feature of volatility dampens the effect of current shocks in volatility on futures price (see Grünbichler and Longstaff (1996, p. 990)).

Actually, the spot VIX and VIX futures do not converge at expiration because of a settlement bias due to the procedure used to determine the final settlement value for the VIX futures. Although the spot VIX is determined based on the average of the bid and ask quotes of the relevant out-of-the-money S&P 500 options, the settlement procedure employs the actual trade prices of the nearby S&P 500 options at the opening on the futures settlement day.\footnote{In addition, the reported spot VIX is based on a weighted average of two near-to-expiration S&P 500 options, whereas the VIX settlement employs only the next S&P 500 option expiration.} Using opening option prices causes the settlement VIX value to move toward either the average bid or the average ask price as VIX futures dealers unwind.
their strip of option positions. However, the size and direction of the difference from the actual spot VIX is uncertain and on average differs from the spot VIX by 0.26 index points ($260 per contract). The consequence of this bias is that the spot VIX and the VIX futures do not converge at expiration. The market will therefore adjust for this characteristic of the VIX futures and price this factor into the futures, especially into the nearby expiration. Pavlova and Daigler (2008) discuss this issue in detail and propose an alternative settlement procedure to mitigate it.
5 Calibrating the VIX futures price model

5.1 Data description

The data set consists of closing market quotes for VIX futures in the period 2011-2012, i.e. 502 trading days. We consider daily data, fetching all available quoted maturities. Daily closings for the spot VIX are collected, as well.\footnote{VIX and VIX futures quotes are available on the Chicago Board Options Exchange website: http://www.cboe.com.} The number of VIX futures quoted maturities per day is not fixed, as one can find in the CFE VIX futures contract specifications.\footnote{The CFE may list for trading up to nine near-term serial months and five months on the February quarterly cycle for the VIX futures contract.} Figure 1 shows the historical behavior of VIX index from 2003, the year in which it was updated, to 2012.

![Figure 1: Historical values of the spot VIX and the SPX (2003-2012)](image)

In the same figure we show S&P 500 (SPX) data to appreciate the negative correlation between VIX and SPX. Figure 2 shows what happened in our sample period, in the same fashion of Figure 1. In Figure 3 the number of quoted VIX futures per day for the period 2011-2012 is provided. For each VIX futures we collect daily closing quotes up to the last trading day, which is one day before the final settlement date of the relevant VIX futures. These dates are computed
Figure 2: Values of the spot VIX and the SPX for the period 2011-2012

Figure 3: Number of VIX futures per day for the period 2011-2012
Table 1: List of VIX futures quoted on 8th June 2012

<table>
<thead>
<tr>
<th>Futures code</th>
<th>Close</th>
<th>Final settlement date</th>
<th>Last trading date</th>
</tr>
</thead>
<tbody>
<tr>
<td>M (Jun 12)</td>
<td>21.71</td>
<td>06/20/12</td>
<td>06/19/12</td>
</tr>
<tr>
<td>N (Jul 12)</td>
<td>23.83</td>
<td>07/18/12</td>
<td>07/17/12</td>
</tr>
<tr>
<td>Q (Aug 12)</td>
<td>25.07</td>
<td>08/22/12</td>
<td>08/21/12</td>
</tr>
<tr>
<td>U (Sep 12)</td>
<td>26.18</td>
<td>09/19/12</td>
<td>09/18/12</td>
</tr>
<tr>
<td>V (Oct 12)</td>
<td>27.16</td>
<td>10/17/12</td>
<td>10/16/12</td>
</tr>
<tr>
<td>X (Nov 12)</td>
<td>27.76</td>
<td>11/21/12</td>
<td>11/20/12</td>
</tr>
<tr>
<td>Z (Dec 12)</td>
<td>27.79</td>
<td>12/19/12</td>
<td>12/18/12</td>
</tr>
<tr>
<td>F (Jan 13)</td>
<td>28.84</td>
<td>01/16/13</td>
<td>01/15/13</td>
</tr>
<tr>
<td>G (Feb 13)</td>
<td>29.50</td>
<td>02/13/13</td>
<td>02/12/13</td>
</tr>
</tbody>
</table>

according to the VIX futures contract specifications. Table 1 provides, as an example, the list of VIX futures quoted on 8th June 2012.\textsuperscript{10}

5.2 Financial variables

We now focus on the financial meaning of variables in equation (21). The spot VIX quote $V_0$ has a clear financial meaning, whereas $\alpha$ and $\beta$ are less intuitive. Performing a change of variable on $\alpha$ and $\beta$ we introduce two new variables with a clear financial meaning, as well. This will ease our work in creating risk simulations. We introduce the following financial variables: $V_\infty = \alpha/\beta$ and $\tau = 1/\beta$. Note that $V_\infty$ is a volatility and $\tau$ is a time scale (i.e. the reciprocal of the speed of mean reversion parameter). Our pricing formula can be written in terms of the new financial variables as

$$F(\tau, V_\infty, V_0; T) = V_0 e^{-\frac{T}{\tau}} + V_\infty (1 - e^{-\frac{T}{\tau}}).$$

We compute the limits for very small and very large maturities

$$\lim_{T \to 0} F(T) = V_0; \lim_{T \to \infty} F(T) = V_\infty.$$ \textsuperscript{(23)}

Therefore the market expectation for the VIX futures are that it is close to the VIX spot price at short maturities and that it reaches the long-term value $V_\infty$ for long maturities. The time scale $\tau$ provides the market expectation of how

\textsuperscript{10}The expiration month of futures is denoted with a capital letter, according to an international convention for futures code.
long it will take to go from $V_0$ to $V_\infty$.

On any given day we observe closing futures quotes $F_k$ for a number $n$ of maturities. We compute $T$, the time-to-expiration of each quoted contract, as the year fraction which should pass to get from the relevant trade date to the VIX futures last trading date. We use the Act/365 Fixed day-count convention (the year is assumed to be 365 days long) to compute $T$. Ultimately, denoting with $d_1$ the relevant trade date and with $d_2$ the last trading date for the relevant VIX futures contract, we compute the year fraction $T$ as

$$T(d_1, d_2) = \frac{N}{365}. \quad (24)$$

where $N$ is the number of calendar days between $d_1$ and $d_2$.

We want to calibrate the value of $V_0$, $\tau$, and $V_\infty$ for each historical date using a least-square approach hence we decide to calibrate also the value of $V_0$, instead of using the spot VIX quote, because of the non-convergence problem of the cash VIX and the VIX futures at expiration. The calibrated values of $V_0$ will allow us to compute a daily measure of this non-convergence problem, i.e. the basis, defined as the difference between the cash VIX and $V_0$. Using a least-square approach to calibrate $V_0$, $\tau$, and $V_\infty$ mathematically means to find values of $V_0$, $\tau$, and $V_\infty$ that minimize the following target function

$$\phi(V_0, \tau, V_\infty) = \sum_{k=1}^{n} (F_k - F(V_0, \tau, V_\infty; T_k))^2. \quad (25)$$

### 5.3 The MatLab® function lsqcurvefit

We decide to carry out the calibration procedure with the software MatLab®, mainly because we are familiar with this numerical computing environment. The possibility to create and manipulate arrays and matrices, jointly with an iterative procedure, will ease our work to arrange and handle the bulk of data. The Optimization Toolbox™ includes several curve fitting functions. The `lsqcurvefit` function best suits our aim, as it solves nonlinear curve-fitting (data-fitting) problems in least-squares sense. We pass as inputs, the pricing function $F(\tau, V_\infty, V_0; T)$ and, for each day, the VIX futures closing quotes $F_k$ as well as the time-to-expiration year fractions $T_k$. The function also requires as inputs the starting values of the variables to calibrate. `lsqcurvefit` squares
and sums $F_k - F(\tau, V_\infty, V_0; T_k)$, as in the left-hand side of (25), and finds the values of $V_0$, $\tau$, and $V_\infty$ to best fit the function $F(\tau, V_\infty, V_0; T)$ to the market quotes $F_k$. Additionally, one can pass a set of lower and upper bounds to define a feasible range for the variables so that the solution is always in that range.

5.4 Trial and error

Before to go on explaining the calibration procedure we employ, it would be appropriate to make some observations about the way we carry on the analysis. The mathematical solution of the target function is the set of values that, assigned to the financial variables, minimizes the left-hand side in (25). Primarily, we aim at reasonable results, consistent from a financial point of view, in order to perform an efficient risk simulation. The order of magnitude and the stability are crucial features for the results to be reasonable and consistent. However, the solution satisfying these requirements may not match the mathematical one. The order of magnitude is easily manageable, e.g. providing ad hoc ranges to the optimization problem, whereas the stability of the parameters may be a complex issue. We want the time path of the parameters to reflect that of actual market data. Too unstable parameters in addition to be disappointed by the real financial market data, prevent the simulation to be reliable. In the end, our analysis is based on a trial and error approach aiming at a reliable risk simulation in a consistent framework.

5.5 The Least-squares-1 procedure

The first attempt to calibrate the model leads to unsatisfactory results. For each day in the data set we run the regular least-square procedure described in 5.3, computing the values of $V_0$, $\tau$, and $V_\infty$. We call this procedure Least-squares-1 for argument’s sake. Figure 4 and Figure 5 exhibit the calibrated values for each financial variable. To run the Least-squares-1 procedure we define the lower and the upper bounds for the variables according to their financial meanings. $V_0$ and $V_\infty$ are volatilities, therefore we set the minimum historical daily closing quote for VIX futures, 11.00, as the lower bound and the maximum historical daily closing quote, 219.00, as the upper bound, in both cases. As far as $\tau$ is concerned, it is a time scale parameter and we let it to vary.
Figure 4: Least-squares-1 - Daily values of $V_0$ and $V_\infty$

Calibration of the VIX futures pricing function: daily values of $V_0$ and $V_\infty$ that result from the regular least-squares procedure.

Figure 5: Least-squares-1 - Daily values of $\tau$

Calibration of the VIX futures pricing function: daily values of $\tau$ that result from the regular least-squares procedure.
from 0 to 5 years. The ranges are wide but we want to put as less restrictions as possible. The main problem one can see in the results is the instability of values, especially for $V_0$ and $\tau$. In some cases the algorithm can not find a solution and provides values that equal the lower or the upper bound, highlighting the instability problem. Provided that the range we set is wide, the fact that the algorithm can not find a solution is not a consequence of tight restrictions. So there may be mathematical issues to investigate. However, there is a positive result supporting this first attempt. As $V_0$ is a short-term volatility measure, we expect it to be more volatile than $V_\infty$, which, in turn, we expect to be somewhat stable, being the long-run mean in the mean-reverting volatility process. The results from Least-squares-1 can account for these features as can be seen in Figure 4.

5.6 A two-step optimization method: Least-squares-2

The most unstable result in Least-squares-1 is about $\tau$. First of all, note that the pricing function $F(\tau, V_\infty, V_0)$ is linear in the variables $V_0$ and $V_\infty$, but is non-linear in $\tau$. This feature may prevent the algorithm to find an efficient solution, maybe because of many degrees of freedom in the minimization problem. We take into account this observation and define a different algorithm, which we call Least-squares-2. This procedure splits the minimization problem in two sub-problems. In the former we search for the optimal values for $V_0$ and $V_\infty$, fixing the value of $\tau$. In the latter we fine-tune the value of $\tau$, in the light of the optimal values for $V_0$ and $V_\infty$ we obtain from the former. Thanks to this shortcut, we can run the minimizations without setting a range for the variables. Figure 6, Figure 7, and Figure 8 show the results of Least-squares-2 for different initial values of $\tau$. The fact that $V_0$ is more volatile than $V_\infty$ is now even more evident, but the initial value of $\tau$ is crucial. There is yet a source of instability and the path of $\tau$ is strongly dependent from the initial value we fix. Therefore this fine-tuning technique does not completely solve the issues of Least-squares-1. To better grasp the mathematical meaning of $\tau$ we show in Figure 9 what happens to the pricing function if we let $\tau$ to change, fixing the values of $V_0$ and $V_\infty$. The example comes from our real market data and regards VIX futures quotes in Table 1, which are plotted as well. Figure 9 serves also as a sketch for the VIX futures term-structure we want to obtain calibrating the model via the least-squares procedure. As one can see, $\tau$ measures the curvature of the
Figure 6: Least-squares-2 - Daily values of $V_0$ for different initial values of $\tau$
Calibration of the VIX futures pricing function: daily values of $V_0$ that result from the two-step optimization approach. Results are provided for different initial values of $\tau$, fixed in the first step of the procedure.

Figure 7: Least-squares-2 - Daily values of $V_\infty$ for different initial values of $\tau$
Calibration of the VIX futures pricing function: daily values of $V_\infty$ that result from the two-step optimization approach. Results are provided for different initial values of $\tau$, fixed in the first step of the procedure.
Figure 8: Least-squares-2 - Fine-tuning of the different initial values of $\tau$
Calibration of the VIX futures pricing function: daily values of $\tau$ that result from the two-step optimization approach. In the second step of the procedure the value of $\tau$ is fine-tuned, according to the results obtained in the first step.

Figure 9: Pricing function behavior for different values of $\tau$
Calibration of the VIX futures pricing function: mathematical meaning of $\tau$. Fixing the values of $V_0$ and $V_\infty$, the pricing function curvature changes as $\tau$ is let to change. High values of $\tau$ correspond to a flat function, whereas low values of $\tau$ correspond to a bended function.
function, i.e. it tells us how much the function is bended or flat. We also recall the financial meaning of \( \tau \), i.e. the market expectation of how long it will take to go from \( V_0 \) to \( V_\infty \), and that is the reciprocal of the speed of mean-reversion \( \beta \). Figure 9 confirms \( \tau \) to be a crucial issue.

5.7 Another two-step optimization method: Least-squares-3

We try to improve further the procedure, still following a two-step optimization approach. Therefore, we refine Least-squares-2 just adding an iterative component to the algorithm. We call this produce Least-squares-3, which is still the combination of two sub-problems. In the former we search for the optimal values for \( V_0 \) and \( V_\infty \), but this time we do not fix the value of \( \tau \), except for the first day. In the latter we fine-tune the value of \( \tau \), in the light of the optimal values for \( V_0 \) and \( V_\infty \) we obtain from the former. In addition, the value that results from the fine-tuning of \( \tau \) is used in the next day to calibrate \( V_0 \) and \( V_\infty \).

We explain briefly: starting from day one fixing \( \tau \), e.g. equal to 0.5, the values of \( V_0 \) and \( V_\infty \) are calibrated, then \( \tau \) is fine-tuned, so we use the fine-tuned \( \tau \) for day two to calibrate the values of \( V_0 \) and \( V_\infty \), then \( \tau \) is fine-tuned, and so on and so forth. To prevent \( \tau \) to reach zero, we add to the iterative procedure an if-then construct that resets \( \tau \) to the value of one week (i.e. \( 7/365 \)) when it becomes less than one day (i.e. a value less than \( 1/365 \)). Least-squares-3 works pretty well and provides excellent results. Once again the behavior of \( \tau \) is the key. In Figure 10 we show the path of \( \tau \) as a result of Least-squares-3 for different initial values fixed on the first day. We find \( \tau \) to level off very quickly. It takes about ten months for different \( \tau \) paths to converge to the same one. Once the different \( \tau \) paths have converged, the initial value of \( \tau \) is no more crucial and, for this reason, deciding what should be the starting value is less binding.

The convergence of \( \tau \) occurs day by day, according to the VIX futures market structure information, which is included in VIX futures quotes. In conclusion, we approve the Least-squares-3 procedure as it provides satisfactory results.

The two-step least-squares procedure let us to calibrate the three financial variables for each day in our sample period. We fix the starting value of \( \tau \) equal to 0.5, i.e. half a year. Figure 11 shows the results. Figure 11 has a remarkable explanatory power, provided that it shows all relevant features of our model. \( V_0 \)
Calibration of the VIX futures pricing function: convergence of different $\tau$ paths. The value of $\tau$ is fixed only for the first day of the sample period and then is let to level off according to the VIX futures market structure information.

Calibration of the VIX futures pricing function: daily values of $V_0$, $V_\infty$, and $\tau$ that result from the two-step optimization approach. The fine-tuned value of $\tau$ is used in the next day to calibrate $V_0$ and $V_\infty$. 

Figure 10: Least-squares-3 - $\tau$ path for different starting values

Figure 11: Calibrated daily values of $V_0$, $V_\infty$, and $\tau$
is clearly more volatile than $V_{\infty}$, which instead is a more stable value, and $V_0$ tends to revert to $V_{\infty}$ when it is above or below it. Low values of $\tau$ correspond to the gap between $V_0$ and $V_{\infty}$ to widen, whereas high values correspond to the gap to tighten. Figure 12 and Figure 13 validate our choice to calibrate the value for $V_0$ instead of using the spot VIX quote as the model would suggest. Indeed, the difference between the spot VIX and $V_0$ is significant. We compute the basis in relative terms as

$$basis = \frac{VIX}{V_0} - 1. \quad (26)$$

Figure 12: $V_0$ daily values and VIX spot closing quotes

*Calibration of the VIX futures pricing function: non-convergence of the VIX index and VIX futures at expiration.* $V_0$ is the extrapolated value from the pricing function for $T = 0$, i.e. the price of a VIX futures with a time-to-expiration equal to zero. So $V_0$ should converge to the VIX index, but the figure shows that there is a significant difference.
The basis lies always in the same range except for the period from August 2011 to October 2011. In August 2011 the S&P 500 index started to decline, the VIX reached very high values, and the VIX futures term structure flipped to backwardation, after having been in contango for more than a year. After this period the term structure shifted again to contango, which is the normal market condition for VIX, and so standed for the remaining part of our sample period. This violation of the VIX normal market condition is useful to test the pricing performance of our model. Indeed, we can study whether our model is able to yield reliable results in a variety of market circumstances.

We use the resultant estimated parameters for each day to determine the fair VIX futures prices. We compare actual VIX futures prices to the estimated fair futures prices and for all futures available in our data set we calculate the

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**Figure 13: Relative basis**

Computation of the basis. The basis is the difference between the spot VIX and $V_0$, computed in relative terms as in equation (26).

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\[11\] The term contango is used to depict the market condition for which futures contracts are traded above the spot price. In such a market condition the futures curve is typically upward sloping. The term backwardation is used to depict the opposite market condition, when futures contracts are traded below the spot price and the futures curve is typically downward sloping.
The term contango is used to depict the market condition for which futures contracts are traded above the spot price. This condition causes the futures curve to be upward sloping. In the figure, the green squares point the VIX futures quotes of the relevant day, the blue dot marks the VIX spot quote, and the green line is the interpolated VIX futures curve which includes also the extrapolated value at zero.

Figure 14: Example of VIX futures term structure in contango

The term contango is used to depict the market condition for which futures contracts are traded above the spot price. This condition causes the futures curve to be upward sloping. In the figure, the green squares point the VIX futures quotes of the relevant day, the blue dot marks the VIX spot quote, and the green line is the interpolated VIX futures curve which includes also the extrapolated value at zero.

absolute percentage error (APE) as

\[
APE = \left| \frac{F - F(V_0, \tau, V_\infty; T)}{F(V_0, \tau, V_\infty; T)} \right|
\]

Errors are determined in basis points of the VIX (one basis point is worth $10 per futures contract). We obtain a mean APE of about 1.035 basis points which is worth $10.35 per futures contract. The error ranges from a minimum of 0 to a maximum of 15.24 basis points ($152.4 per futures contract).

Given the estimated values of the parameters we can plot the VIX futures term structure for each day in our sample period. Figure 14 shows a typical
contango VIX futures term structure. In Figure 14 the green squares point the VIX futures quotes of the relevant day, the VIX spot quote is marked with the blue dot, and the green line is the interpolated VIX futures curve. The curve includes also the extrapolated value at zero, i.e. the value of $V_0$, so that the basis is observable.

5.8 The “Christmas effect”

Figure 15 shows VIX futures term structure in backwardation and the so-called “Christmas effect”. The “Christmas effect” is a VIX calendar effect, well-known among practitioners, and denotes the tendency for VIX to drop down to relatively low levels during the Christmas holidays. VIX futures market would predict this drop for months in advance and as a consequence the December contract (magenta diamond) would be traded at a lower price than November and January contracts. The difference would be at least of 1-2 basis points, which is just how much we would need to lift December to straighten up the VIX term structure in Figure 14 into a normal backwardation. In the calibration procedure this effect causes the curve (magenta) to shift down to interpolate the December contract. However, this effect is not always strikingly notable. For example, in Figure 16 we provide the term structure on 22nd August 2012, which is exactly one year after that of Figure 15, and, as one can see, the effect is much less evident. At the best of our knowledge the academia has not dealt with this issue so far. We believe that is necessary to investigate it with more data in hand and in case to include factors to account for seasonality as in the commodity futures market.
The term backwardation is used to depict the market condition for which futures contracts are traded below the spot price. This condition causes the futures curve to be downward sloping.

The “Christmas effect” causes the VIX futures curve (magenta) to shift down to interpolate the December contract (magenta diamond), which is traded at a lower price than November and January contracts.
The “Christmas effect” seems to be much less evident when the VIX futures term structure is in contango.
6 Risk simulations

6.1 The historical-simulation model

We want to perform numerical simulations to assess the risk of VIX futures. Given an asset, for which we have a model and a pricing function, we can identify the risk factors affecting its value, i.e. the financial variables from which we expect the risk to come from. In the case of a simple asset, such as a stock, the only risk factor is the stock price. In the case of derivative securities or complex financial instruments we can express the price as a function of some financial variables, hence we have more risk factors to consider. The basic idea behind a risk simulation is to compute the expected distribution of the risk factors affecting the value of an asset and use this distribution to compute risk measures, i.e. to assess the risk of that asset. Our approach is the historical-simulation model, which does not make any analytical assumption on the expected distribution of the risk factors and extracts risk figures from a discrete number of scenarios. The only assumption we make is that the risk factors distribution is stationary. In other words we assume that the simulated values of the risk factors are distributed like the historical ones. Given a generic asset with price \( P_t \) at the current time \( t \), the historical simulation procedure starts defining a discrete distribution of \( N \) possible price scenarios \( P_i, i = 1, \ldots, N \), for the given asset price \( P_T \) at time \( T \) in the future with \( T > t \). The \( N \) scenarios give the expected distribution of the asset price \( P_T \). If the relevant asset is always expected to have a positive price, we can build multiplicative-type scenarios from \( N \) positive numbers \( S_i \) as

\[
P_i = P_t S_i.
\]

(28)

for \( i = 1, \ldots, N \). The simulation procedure we adopt for VIX futures is described hereafter in detail.
6.2 VIX futures scenarios

We want to create historically-simulated scenarios of a VIX futures with maturity $T$ and a market quote $F_q$. The latest date is set as the reference date as well as the latest values of the financial variables are set as reference values, $V_0^r$, $\tau_r$, and $V_\infty^r$. Using the reference values for the risk factors we can compute the futures model price for each maturity $T$ as

$$F^r = V_0^r e^{-\frac{T}{\tau_r}} + V_\infty^r (1 - e^{-\frac{T}{\tau_r}}).$$

The VIX futures price is function of three financial variables, that are the risk factors. As the financial variables are always positive we can create multiplicative-type historical scenarios for each financial variable as

$$V_0^{sim[i]} = V_0^r \frac{V_0^i}{V_0^{i-1}}.$$  \hspace{1cm} (30)

$$V_\infty^{sim[i]} = V_\infty^r \frac{V_\infty^i}{V_\infty^{i-1}}.$$  \hspace{1cm} (31)

$$\tau^{sim[i]} = \tau_r \frac{\tau_i}{\tau^{i-1}}.$$  \hspace{1cm} (32)

where $i = 1, ..., N$ and $N$ is the number of scenarios. These scenarios are used to compute the simulated model-price scenarios, i.e.

$$F^{sim[i]} = V_0^{sim[i]} e^{-\frac{T}{\tau^{sim[i]}}} + V_\infty^{sim[i]} (1 - e^{-\frac{T}{\tau^{sim[i]}}}).$$

that can be applied to the quotes on the reference date to obtain the simulated quotes scenarios

$$F_{q}^{sim[i]} = F_q \frac{F^{sim[i]}}{F^r}.$$  \hspace{1cm} (34)

The distribution of $\{F_{q}^{sim[i]}\}$ is then used to compute risk figures.

6.3 VIX futures spread scenarios

We now apply the simulation strategy described in subsection 6.2 to a calendar spread position on VIX futures. A calendar spread on futures is a spread trade involving the simultaneous purchase and sale of futures expiring at different dates. The usual case involves a long position on futures with a longer
maturity and a short position on futures with a nearer maturity. To have a positive value this synthetic position requires the futures market to be in contango. The opposite, i.e. a long position on futures with a nearer maturity and a short position on futures with a longer maturity, is called reverse calendar spread and a futures market in backwardation is required for this position to have a positive value. Indeed, the latter is the typical spread trade on commodity futures. We choose to simulate the risk of the former as the normal market condition for VIX futures is to be in contango, and this should ensure a positive value. However, there is a small but positive probability for the spread to become negative at a future date. Let $T_1$ and $F^q_1$ be the maturity and the market quote of the VIX futures to sell, and $T_2$ and $F^q_2$ the maturity and the market quote of the VIX futures to buy, with $T_2 > T_1$, we want to create simulated quotes scenarios of

$$CS^q = F^q_2 - F^q_1.$$  \hspace{1cm} (35)$$

where $CS^q$ denotes the value of the calendar spread. Using the reference values for the risk factors we can compute the futures model price for maturities $T_1$ and $T_2$ as

$$F^r_1 = V^r_0 e^{-\frac{T_1}{\tau}} + V^r_{\infty}(1 - e^{-\frac{T_1}{\tau}}).$$ \hspace{1cm} (36)$$

$$F^r_2 = V^r_0 e^{-\frac{T_2}{\tau}} + V^r_{\infty}(1 - e^{-\frac{T_2}{\tau}}).$$ \hspace{1cm} (37)$$

We proceed similarly for the financial-variable scenarios, so, with $i = 1, ..., N$, the simulated model-price scenarios are

$$F^1_{sim[i]} = V^0_{sim[i]} e^{-\frac{T_1}{\tau_{sim[i]}}} + V^r_{\infty}(1 - e^{-\frac{T_1}{\tau_{sim[i]}}}).$$ \hspace{1cm} (38)$$

$$F^2_{sim[i]} = V^0_{sim[i]} e^{-\frac{T_2}{\tau_{sim[i]}}} + V^r_{\infty}(1 - e^{-\frac{T_2}{\tau_{sim[i]}}}).$$ \hspace{1cm} (39)$$

that can be applied to the quotes on the reference date to obtain the simulated quotes scenarios

$$F^1_{q,1[i]} = F^q_1 \frac{F^1_{sim[i]}}{F^r_1}. \hspace{1cm} (40)$$

$$F^2_{q,2[i]} = F^q_2 \frac{F^2_{sim[i]}}{F^r_2}. \hspace{1cm} (41)$$

Finally we compute the simulated spread value scenarios as

$$CS^q_{sim[i]} = F^q_{q,2[i]} - F^q_{q,1[i]}.$$ \hspace{1cm} (42)$$
Table 2: VIX futures calendar spread (reference date 12/31/2012)

*Computation of the VIX futures spread value on the reference date, 31st December 2012. The spread value (CS\(^{q}\)) is the difference between the quote of the VIX futures to buy, expiring September 2013 (\(F_{2}^{q}\)), and the quote of the VIX futures to sell, expiring March 2013 (\(F_{1}^{q}\)).*

<table>
<thead>
<tr>
<th>Contract</th>
<th>Last trading date</th>
<th>Year fraction</th>
<th>Quote</th>
<th>Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar 2013</td>
<td>03/19/13</td>
<td>0.2137</td>
<td>19.58</td>
<td></td>
</tr>
<tr>
<td>Sep 2013</td>
<td>09/17/13</td>
<td>0.7123</td>
<td>23.52</td>
<td>3.94</td>
</tr>
</tbody>
</table>

Table 3: Reference values and VIX futures model prices

*Computation of the VIX futures model prices for maturities \(T_{1}\) and \(T_{2}\) using the reference values for the risk factors.*

| \(V_{0}\)  | 16.842 | \(P_{1}^{r}\) | 19.64 |
|\(V_{\infty}\) | 26.778 | \(P_{2}^{r}\) | 23.48 |
|\(\tau_{r}\) | 0.6454 |            |       |

We consider now the historically-simulated scenarios for a calendar spread made up of the VIX futures contracts in Table 2. We create \(N = 501\) equiprobable daily scenarios, setting as reference date the last date of our sample, i.e., 31st December 2012. Table 3 shows the reference values of the financial variables and the futures model prices computed on the reference date. In Table 4 we show the financial-variable scenarios (for brevity scenarios for \(i = 6, ..., 496\) have been omitted). Finally, in Table 5 we provide the simulation results. The distribution outcome of \(\{CS_{q}^{\text{sim}[i]}\}\) is used to compute risk figures. To convert the simulated spread values in cash is sufficient to multiply each \(CS_{q}^{\text{sim}[i]}\) by $1000, which is the contract multiplier for VIX futures according to the CFE rule. In the following section we go on considering the simulated spread values as VIX index points for convenience’s sake. Of course nothing changes if one considers the dollar amounts corresponding to each scenario.
Table 4: Financial variables scenarios

Computation of financial variables scenarios. For brevity scenarios for $i = 6, \ldots, 496$ have been omitted.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$V^i_0$</th>
<th>$V^i_{\infty}$</th>
<th>$\tau^i_0$</th>
<th>$V^i_{\text{sim}}(\tau_0)$</th>
<th>$V^i_{\text{sim}}(\tau_{\infty})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.842</td>
<td>26.778</td>
<td>0.6454</td>
<td>13.357</td>
<td>28.901</td>
</tr>
<tr>
<td>2</td>
<td>21.237</td>
<td>24.811</td>
<td>0.6430</td>
<td>19.943</td>
<td>26.117</td>
</tr>
<tr>
<td>3</td>
<td>17.935</td>
<td>25.439</td>
<td>0.6148</td>
<td>16.660</td>
<td>26.469</td>
</tr>
<tr>
<td>4</td>
<td>18.131</td>
<td>25.736</td>
<td>0.6061</td>
<td>17.630</td>
<td>26.973</td>
</tr>
<tr>
<td>5</td>
<td>17.321</td>
<td>25.550</td>
<td>0.5970</td>
<td>17.417</td>
<td>26.496</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>497</td>
<td>18.799</td>
<td>29.242</td>
<td>0.4558</td>
<td>16.891</td>
<td>26.856</td>
</tr>
<tr>
<td>498</td>
<td>18.745</td>
<td>29.157</td>
<td>0.4621</td>
<td>16.911</td>
<td>26.835</td>
</tr>
<tr>
<td>499</td>
<td>18.669</td>
<td>29.095</td>
<td>0.4695</td>
<td>16.829</td>
<td>26.683</td>
</tr>
<tr>
<td>500</td>
<td>18.683</td>
<td>29.199</td>
<td>0.4771</td>
<td>16.504</td>
<td>26.507</td>
</tr>
<tr>
<td>501</td>
<td>19.066</td>
<td>29.498</td>
<td>0.4847</td>
<td>16.753</td>
<td>26.509</td>
</tr>
</tbody>
</table>

Table 5: Simulated spread scenarios

Computation of the simulated model-price scenarios and the simulated quotes scenarios for maturities $T_1$ and $T_2$. The simulated spread scenarios are computed as in equation (42). For brevity scenarios for $i = 6, \ldots, 496$ have been omitted.

<table>
<thead>
<tr>
<th>Model-price scenarios</th>
<th>Quotes scenarios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mar 2013</td>
</tr>
<tr>
<td>$i$</td>
<td>$F^i_{\text{sim}}(T_1)$</td>
</tr>
<tr>
<td>1</td>
<td>17.72</td>
</tr>
<tr>
<td>2</td>
<td>21.62</td>
</tr>
<tr>
<td>3</td>
<td>19.39</td>
</tr>
<tr>
<td>4</td>
<td>20.23</td>
</tr>
<tr>
<td>5</td>
<td>19.95</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>497</td>
<td>19.73</td>
</tr>
<tr>
<td>498</td>
<td>19.75</td>
</tr>
<tr>
<td>499</td>
<td>19.64</td>
</tr>
<tr>
<td>500</td>
<td>19.36</td>
</tr>
<tr>
<td>501</td>
<td>19.54</td>
</tr>
</tbody>
</table>
The VIX futures spread values are computed as in equation (42). It is also shown the P&L\textsuperscript{i} computed as in equation (43).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
i & \( CS_q^{sim[i]} \) & \( P&L^i \) & \( P&L_{75}\%^i \) \\
\hline
1 & 10.94 & 1.7767 & 177.67\% \\
2 & 2.46 & -0.3765 & -37.65\% \\
3 & 3.87 & -0.0173 & -1.73\% \\
4 & 3.70 & -0.0622 & -6.22\% \\
5 & 3.60 & -0.0870 & -8.70\% \\
\vdots & \vdots & \vdots & \vdots \\
497 & 3.97 & 0.0074 & 0.74\% \\
498 & 3.96 & 0.0041 & 0.41\% \\
499 & 3.93 & -0.0030 & -0.30\% \\
500 & 3.98 & 0.0113 & 1.13\% \\
501 & 3.89 & -0.0128 & -1.28\% \\
\hline
\end{tabular}
\caption{VIX futures spread values and P&L strip}
\end{table}

6.4 Risk figures

From the distribution of \( \{CS_q^{sim[i]}\} \) we derive the P&L (profit and loss) strip of the calendar spread position on VIX futures as

\[ P&L^i = \frac{CS_q^{sim[i]}}{CS_q} - 1. \]  \hfill (43)

Table 6 shows the P&L strip while Table 7 shows the sorted P&L strip from the worst to the best case.

First of all we compute the mean P&L and its standard deviation. The mean P&L is -0.78\% and the standard deviation is 13.57\%. This means that we expect on average a 0.78\% loss with a standard deviation of 13.57\%. Since the actual distribution of \( \{CS_q^{sim[i]}\} \) is unknown we can use these values only as a very rough idea of what the actual average loss would be. We prefer, instead, to compute other statistical quantities that better gauge the risk of loss, such as downside deviation and Value at Risk.

We now focus on more accurate risk measures that allow to make distinction between the risk to suffer losses, i.e. the downside risk, and the chance to get gains, i.e. the upside potential. The most popular measure of downside risk is the semideviation, that is a dispersion measure of returns falling below the
Table 7: Sorted VIX futures spread values and P&L strip

VIX futures spread values in increasing order from the smallest to the largest. It is also shown the sorted P&L strip.

<table>
<thead>
<tr>
<th>i</th>
<th>CS^sm[t]</th>
<th>P&amp;L</th>
<th>P&amp;L^{1/2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>304</td>
<td>1.48</td>
<td>-0.6243</td>
<td>-62.43%</td>
</tr>
<tr>
<td>277</td>
<td>1.53</td>
<td>-0.617</td>
<td>-61.17%</td>
</tr>
<tr>
<td>331</td>
<td>1.93</td>
<td>-0.5112</td>
<td>-51.12%</td>
</tr>
<tr>
<td>352</td>
<td>1.95</td>
<td>-0.5061</td>
<td>-50.61%</td>
</tr>
<tr>
<td>296</td>
<td>2.01</td>
<td>-0.4897</td>
<td>-48.97%</td>
</tr>
<tr>
<td>346</td>
<td>4.99</td>
<td>0.2676</td>
<td>26.76%</td>
</tr>
<tr>
<td>154</td>
<td>5.05</td>
<td>0.2807</td>
<td>28.07%</td>
</tr>
<tr>
<td>328</td>
<td>5.27</td>
<td>0.3380</td>
<td>33.80%</td>
</tr>
<tr>
<td>351</td>
<td>5.70</td>
<td>0.4473</td>
<td>44.73%</td>
</tr>
<tr>
<td>1</td>
<td>10.94</td>
<td>1.7767</td>
<td>177.67%</td>
</tr>
</tbody>
</table>

observed mean return, and computed as

$$\sigma_d = \sqrt{\frac{1}{n} \sum_{R_i < \bar{R}} (R_i - \bar{R})^2}.$$  \hspace{1cm} (44)

where the subscript \(d\) means downside and \(n\) is the number of returns below the mean return. If the mean return is replaced by a threshold return \((k)\), usually the minimum acceptable return for the investor, we have the downside deviation \((\sigma^-_d)\). In principle, one can compute the semideviation also for returns higher than the mean return, i.e. the positive semideviation

$$\sigma_u = \sqrt{\frac{1}{n} \sum_{R_i > \bar{R}} (R_i - \bar{R})^2}.$$  \hspace{1cm} (45)

where the subscript \(u\) means upside and \(n\) is the number of returns above the mean return. Again, one can replace the mean return with a threshold return and obtain the upside deviation \((\sigma^+_u)\). The upside potential measures the mean excess return over a threshold return

$$UP = \frac{1}{n} \sum_{R_i > k} (R_i - k).$$  \hspace{1cm} (46)
We compute these risk measures, and, where required, we decide to consider one basis point as threshold return. Results are provided in Table 8. The semideviation and the downside deviation have similar values and we can note the same for the positive semideviation and the upside deviation. According to the upside potential we expect a mean excess return of 7% over the threshold return of one basis point.

The Value at Risk (VaR for short) is a widely used risk measure to assess the risk of loss of a specific asset. Given a certain percentile $p\%$ the value at risk of an asset is defined as the maximum possible loss in the value of that asset within a given time span and at a given confidence level $p\%$. Given a distribution of $N$ observed returns, sorted from the smallest to the largest, to compute the VaR $p\%$, i.e. the loss with a $p\%$ confidence level, we need to consider the lowest loss of the worst $n = (1 - p\%)N$ observations. For example, take the sorted distribution of the VIX futures spread P&Ls and assume for simplicity that it is made of $N = 500$ scenarios. To compute the VaR 99%, i.e. the loss with a 99% confidence level, we need to consider the lowest loss of the worst $n = (1 - 99\%)500 = 5$ scenarios. In other words, the best observation of the worst 5 scenarios is the VaR. In our case it corresponds to the scenario tagged with $i = 296$, i.e. to a spread value of 2.01 and to a percentage loss of 48.97%. The VaR 95% is the best observation of the worst $n = (1 - 95\%)500 = 25$ scenarios, that is the $i = 293$ scenario which corresponds to a spread value of 3.21 and to a percentage loss of 18.57%. The VaR gives rise to two specular explanations: we can say that there is a $1 - p\%$ probability to lose at least the value of the VaR over the relevant horizon or that we are $p\%$ confident the loss will be no greater than the VaR over the same time horizon. The expected shortfall (ES for short) or conditional VaR (CVaR) for a certain percentile $p\%$, is the average loss expected in all but the best $p\%$ of the cases. Basically, the ES $p\%$ accounts for the possibility to incur losses greater than the VaR $p\%$ and it is obtained by averaging the observations comprised within the worst $1 - p\%$ percentile. Therefore, to compute the ES 99% we average the observations comprised within the worst 1% percentile. So in our example the ES 99% is the average of the worst five values (assuming $N = 500$ scenarios for simplicity). Let $\lambda_i$ be the loss of the $i$-th scenario, the ES 99% is

$$ES^{99\%}(CS) = \frac{\lambda_{304} + \lambda_{277} + \lambda_{331} + \lambda_{352} + \lambda_{296}}{5} = 54.86\%.$$  

(47)

Following the same argument, the ES 95% is the average of the worst twenty-five
Table 8: Risk figures (%)  
Risk figures (%) computed from the \( \{ P\&L \} \) distribution with \( N = 501 \) scenarios. \( \text{VaR}^{99\%} \) and \( \text{ES}^{99\%} \) are the result of a linear interpolation according to equation (48). We set the threshold return \( k \) equal to one basis point for the risk measures that require it.

<table>
<thead>
<tr>
<th>( P&amp;L% )</th>
<th>-0.78%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>13.57%</td>
</tr>
<tr>
<td>( \sigma_d )</td>
<td>12.72%</td>
</tr>
<tr>
<td>( \sigma_{k=0.01} )</td>
<td>12.82%</td>
</tr>
<tr>
<td>( \sigma_u )</td>
<td>14.30%</td>
</tr>
<tr>
<td>( \sigma_{k=0.01}^{-} )</td>
<td>14.74%</td>
</tr>
<tr>
<td>( UP_{k=0.01} )</td>
<td>7.00%</td>
</tr>
<tr>
<td>( \text{VaR}^{99%} )</td>
<td>48.97%</td>
</tr>
<tr>
<td>( \text{ES}^{99%} )</td>
<td>54.85%</td>
</tr>
<tr>
<td>( \text{VaR}^{95%} )</td>
<td>18.55%</td>
</tr>
<tr>
<td>( \text{ES}^{95%} )</td>
<td>31.56%</td>
</tr>
</tbody>
</table>

values. So far we have assumed to have a P&L distribution made of \( N = 500 \) scenarios. Actually we have 501 scenarios. So, as a consequence, \( n = 5.01 \) for a confidence level of 99\%, and \( n = 25.05 \) for a confidence level of 95\%. In the computation of value at risk and expected shortfall usually occurs that the index \( n = (1 - p\%)N \) is not an integer. In this case value at risk and expected shortfall are computed for \( n^- \), the closest integer smaller than \( n \), and for \( n^+ \), the closest integer higher than \( n \). The value of the risk figure for \( n \) is then linearly interpolated between the two computed values, according to this general formula

\[
\text{VaR}_n^{p\%} = \frac{n_n^+ - n^-}{n^+ - n^-} \text{VaR}_{n^-} - \frac{n_n^- - n^+}{n^- - n^+} \text{VaR}_{n^+}.
\]  

(48)

which of course holds true for expected shortfall, too. The values of VaR and ES, provided in Table 8, are the results of the application of (48).

First of all we note that the VIX futures spread value never becomes negative in the simulation we performed. According to our estimate, we would never lose the whole (or more than) invested money, provided that a loss of 100\% (or more than) never happens in the different 501 scenarios. The worst case scenario involves a loss of 62.43\%, while the best case scenario, excluding the abnormal return of 177.67\%, commands a gain of 44.73\%. The \( \text{VaR}^{99\%} \) tells us that there is a probability of 1\% to lose at least the 48.97\% of the investment and that, with a probability of 99\% the loss will be no greater than 48.97\%. In the 1\% worst cases we expect an average loss of 54.85\%. The \( \text{VaR}^{95\%} \) tells us that there
is a probability of 5% to incur a loss of at least 18.55% and there is a confidence of 95% that the loss will be no greater than 18.55%. In the 5% worst cases we expect an average loss of 31.56%. Provided that these risk figures have been computed from daily scenarios, their time span is one day as well. We consider the investment in the VIX futures spread rather risky, as with a probability of 1% we expect an average daily loss of more than a half of the invested money, and with a probability of 5% we expect an average daily loss of nearly one third of the invested money. Moreover, the mean P&L is negative, which means that on average we expect to lose money, and the standard deviation is pretty high, considering a negative mean return. This would seem to confirm our consideration of VIX futures spread to be a risky investment.
7 Summary and conclusions

We describe the procedure to carry out risk simulations of VIX futures. We assume a stochastic-volatility CEV process to model the spot VIX dynamics and evaluate the VIX futures price that stems from the CEV model. According to the model, the VIX futures price is an exponentially weighted average of the spot VIX ($V_0$) and the long-run mean of the mean-reverting process ($V_\infty$). Nevertheless, in the calibration of the pricing function, we decided to estimate the value of $V_0$ instead of using the spot VIX quotes. The reason behind this choice is the non-convergence of the VIX index and the VIX futures at expiration. Since we find a significant basis, i.e. the difference between the spot VIX and the estimated value of $V_0$, our choice is supported. The bigger issue in the calibration procedure is the determination of $\tau$, the time scale parameter providing the market expectation of how long it will take to go from $V_0$ to $V_\infty$. To solve for the instability of $\tau$ time path, we first define a two-step least-squares procedure. In this way we deal with $V_0$ and $V_\infty$ separately from $\tau$, taking into account the VIX futures pricing function pattern, which is linear in $V_0$ and $V_\infty$, but non-linear in $\tau$. We then add an iterative component to the algorithm which let $\tau$ to level off day by day, according to VIX futures market structure information, included in VIX futures quotes, which we use to estimate $V_0$ and $V_\infty$. Once we have the time series of the three VIX futures risk factors, $V_0$, $\tau$, and $V_\infty$, we numerically simulate their value in a historical simulation framework, creating daily scenarios. We apply the financial variables scenarios to the computation of the expected distribution of the value of a VIX futures spread. Then we derive the simulated spread P&L (profit and loss) distribution, which is used to compute several risk figures. We examine the risk figures, concluding that the investment in a VIX futures spread should be considered risky.

The stochastic volatility mean-reverting CEV process and the resulting futures pricing formula would seem to be the correct theoretical framework to model the VIX market. The two-step least-square procedure we apply to calibrate the VIX futures pricing function provides interesting results. Comparing actual VIX futures prices to the estimated fair futures prices, we obtain a mean absolute percentage error of about one basis point, with the bulk of the errors ranging from about one to four basis points. These are great results, provided that the typical bid-ask spread for the VIX futures contract lies in a wider range. The fact that our estimates can account for the basis allows to shrink
further the pricing error. Indeed, the non-convergence of the VIX index and the VIX futures at expiration is due to a settlement procedure bias and the market adjust for this characteristic, pricing this factor into the futures. Discarding this feature, would produce greater pricing errors especially for the futures with nearby expiration. A desirable characteristic for a model is to perform good in a variety of market circumstances. Normal market circumstances for the VIX corresponds to the VIX futures to be in contango with the basis being in the normal range. Things are different in periods of market turmoil. For example, as a consequence of the August 2011 S&P 500 decline, the VIX index experienced a sudden increase and the VIX futures curve flipped to backwardation, widening the typical basis. Our model seems to work irrespective of the market circumstances and the overall pricing performance is pretty good. So far, we can support the inclusion of a jump component in the stochastic volatility process to be not necessary, as far as VIX futures pricing is concerned. In the calibration of the VIX futures term structure we find evidence of the “Christmas effect”, addressing it as a future research topic, to better investigate the possible presence of seasonality in the VIX futures market. In conclusion, we can consider our risk historical simulation of VIX futures spreads efficient and reliable, as it is performed in a proper theoretical framework.
References


